2011 Day 2 Question 3A (GPP)

a. $\frac{\partial n}{\partial t} + \nabla \cdot (nv) = 0$ $\qquad$ $\dot{n} + n_0 v' = 0$

Linearize, assume 1D $\longrightarrow$

$\nabla^2 \phi = -4\pi q n$ $\qquad$ $\phi'' = -4\pi q n$

$m \frac{dv}{dt} = -q \nabla \phi$ $\qquad$ $m\dot{v} = -q\phi'$

$\Rightarrow \quad m\dot{v}' = -q\phi''$

$-\frac{m}{n_0}\ddot{n} = 4\pi q^2 n \Rightarrow \left(\frac{4\pi q^2 n_0}{m} + \partial_t^2 \right) n = 0$

Fourier Transform in time: $\quad (\omega_p^2 - \omega^2)\, n = 0 . \quad \omega^2 = \omega_p^2$

b. $E' = -4\pi q n$ $\quad , \quad E = E_0 \cos(kx - \omega_p t)$

$-k E_0 \sin(kx - \omega_p t) = -4\pi q n \rightarrow n(x, t=0) = \frac{k}{4\pi q} E_0 \sin(kx)$

$m\dot{v} = -qE$

$\dot{v} = \frac{q}{m} E_0 \cos(kx - \omega_p t)$

$v = -\frac{q\omega_p}{m} E_0 \sin(kx - \omega_p t) \Big|_{t=0} = -\frac{q\omega_p}{m} E_0 \sin(kx)$

c. If $E(t \to \infty)$ is nonzero, it has completed the damping of f.

$N_{accel} \approx \Delta v\, f_0\left(\frac{\omega}{k} - \Delta v\right)$, $\quad N_{decel} \approx \Delta v\, f_0\left(\frac{\omega}{k} + \Delta v\right)$

$f_0\left(\frac{\omega}{k} \pm \Delta v\right) \approx f_0\left(\frac{\omega}{k}\right) \pm \Delta v\, f_0'\left(\frac{\omega}{k}\right)$

$N = N_a - N_d \approx \Delta v \left[f_0\left(\frac{\omega}{k} - \Delta v\right) - f_0\left(\frac{\omega}{k} + \Delta v\right) \right] = 2\Delta v^2 \left| f_0'\left(\frac{\omega}{k}\right) \right|$

Energy gained $\approx \frac{1}{2} m \left(\left(\frac{\omega}{k}\right)^2 - \left(\frac{\omega}{k} - \Delta v\right)^2 \right) \approx m \left(\frac{\omega}{k}\right) \Delta v$

Total energy transferred $\approx N \cdot E_{gained} = 2m\left(\frac{\omega}{k}\right) \left| f_0'\left(\frac{\omega}{k}\right) \right| \Delta v^3$

(why?) where $\Delta v = v_{tr} = \sqrt{2E_0/mk}$

To not extinguish wave, must have $\frac{E^2}{8\pi} > 2m\left(\frac{\omega}{k}\right) \left| f_0'\left(\frac{\omega}{k}\right) \right| \Delta v_t^3$

[Missing factor of n_0 on RHS. If had, could get $\omega_p^2 \omega \approx \omega_p^3$ and have result.]

2011 Day 2 Q3A (GPP)

a. $\frac{\partial n}{\partial t} + \nabla \cdot (n v) = 0 \longrightarrow \frac{\partial \tilde{n}}{\partial t} + n_0 \nabla \cdot \tilde{v} = 0$

$m \frac{dv}{dt} = -q \nabla \phi \qquad m \frac{\partial \tilde{v}}{\partial t} = -q \nabla \tilde{\phi}$

$\nabla^2 \phi = -4\pi q n \qquad \nabla^2 \tilde{\phi} = -4\pi q \tilde{n}$

$m \frac{\partial}{\partial t}(\nabla \cdot \tilde{v}) = -q \nabla^2 \phi = 4\pi q^2 \tilde{n}$

$m \frac{\partial}{\partial t}\left(-\frac{1}{n_0} \frac{\partial \tilde{n}}{\partial t}\right) = 4\pi q^2 \tilde{n}$

$(\partial_t^2 + \omega_p^2)\tilde{n} = 0, \quad \omega_p^2 = \frac{4\pi n e^2}{m}$

$(-\omega^2 + \omega_p^2) = 0 \rightarrow \omega^2 = \omega_p^2$

b. $\tilde{E} = E_0 \cos(kx - \omega_p t)$, $\cancel{\phi = -\frac{\partial E}{\partial x} = k E_0 \sin(kx + \omega_p t)}$

$\nabla^2 \tilde{\phi} = -4\pi q \tilde{n}$

$-\nabla \cdot \tilde{E} = -4\pi q \tilde{n}$

$k E_0 \sin(kx - \omega_p t) = -4\pi q \tilde{n}(x,t)$

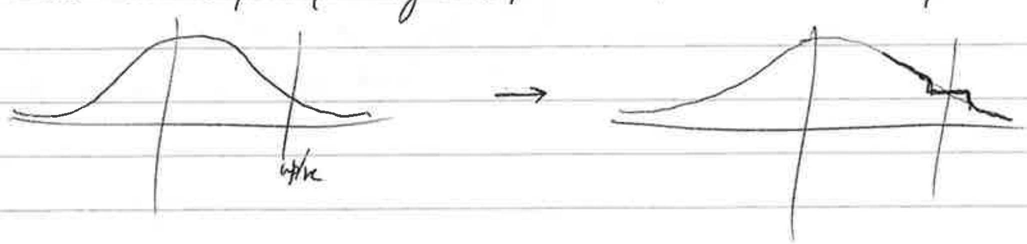
$\tilde{n}(x, t=0) = -k \frac{E_0}{4\pi q} \sin(kx)$

$m \frac{\partial \tilde{v}}{\partial t} = -q \nabla \tilde{\phi} = q \tilde{E} \Rightarrow \frac{\partial \tilde{v}}{\partial t} = \frac{q}{m} E_0 \cos(kx - \omega_p t)$

$\tilde{v}(x,t) = -\frac{q}{m\omega_p} E_0 \sin(kx - \omega_p t), \quad \tilde{v}(x, t=0) = \frac{-qE_0}{m\omega_p} \sin(kx)$

c. If wave not completely extinguished, the distribution will be fully flattened near resonance.

Also damps at ~~$\omega = -\omega_p$?~~



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suppose x already integrated out

$v_p \approx \frac{\omega_p}{k}$

d. $N_{accel} \approx \Delta v\, f_0(v_p - \Delta v)$

$N_{decel} \approx \Delta v\, f_0(v_p + \Delta v)$

$N = N_{accel} - N_{decel} = \Delta v\left(f_0(v_p - \Delta v) - f_0(v_p + \Delta v)\right)$

Assume resonance near tail of distribution, f slowly varying

$f_0(v_p \pm \Delta v) \approx f_0(v_p) \pm \Delta v\, f_0'(v_p) = f_0(v_p) \mp \Delta v\,|f_0'(v_p)|$

$N = 2(\Delta v)^2 |f_0'(v_p)|$

Change in energy for single particle accelerated from $v_p - \Delta v$ to v_p:

$\Delta E = \frac{1}{2} m_e \left(v_p^2 - (v_p - \Delta v)^2\right) = m_e v_p \Delta v$

~~Work done by the wave~~ $\cancel{W = 2\int \vec{E}\cdot d\vec{x} \approx \frac{E_0 e}{k}}$

~~Assume~~ $\cancel{W \approx \Delta E \longrightarrow \Delta v = \frac{E_0 e}{m_e k v_p} = \frac{e E_0}{m_e \omega_p} \equiv V_{trapped}}$

~~To not extinguish wave, must have initial wave energy > total work done:~~

$\cancel{\frac{E_0^2}{4\pi} > N\Delta E = 2 m_e v_p (\Delta v)^3 |f'| = 2 m_e \frac{\omega_p}{k}}$

In frame at phase velocity, $m\ddot{x} = -eE_0 \cos(kx)$

$\rightarrow \ddot{x} + \frac{eE_0 k}{m} x = 0 \rightarrow \Omega_B = \sqrt{\frac{eE_0 k}{m}}$, $V_{tr} = \frac{\Omega_B}{k} = \sqrt{\frac{eE_0}{mk}}$

And assume $\Delta V \approx V_{tr}$.

$$\frac{E_0^2}{4\pi} > N\Delta E = 2 m_e \frac{\omega_p}{k} V_{tr}^3 \left|\frac{\partial f}{\partial v}\left(\frac{\omega_p}{k}\right)\right|$$

$$\frac{1}{4\pi}\left(\frac{mk V_{tr}^2}{e}\right)^2 > 2 m \frac{\omega_p}{k} V_{tr}^3 \left|\frac{\partial f}{\partial v}\right|$$

$$V_{tr} > 2\cdot\frac{4\pi e^2}{m}\frac{\omega_p}{k^3}\left|\frac{\partial f}{\partial v}\right|$$

almost right...

ω_p^2 if there was factor of density...