2011 Day 1 Question 1B (Asymptotics)

a. $\psi'' + Q(x)\psi = 0$, $\psi \sim e^S$

$(S')^2 + S'' + Q(x) \sim 0$ assume $(S')^2 \gg S''$

$S' \sim \pm i\sqrt{Q(x)} \to S \sim \pm i \int \sqrt{Q(x')}\,dx'$

$S'' \sim \pm \frac{1}{2} i\, Q(x)^{-1/2} Q'$ $\qquad Q(x)^{-1/2} Q'(x) \ll 1$

$\left|\frac{S''}{S'^2}\right| = \left|\frac{-Q(x)}{\pm\frac{1}{2}i\, Q^{1/2}}\right|^{-1} \sim \frac{Q'}{Q^{1/2}(x)Q}$ ~~will only be small far from zeros of $Q(x)$.~~

To second order,

$(f'' + g'') + f'^2 + g'^2 + 2f'g' + Q(x) \sim 0$

$\pm\frac{1}{2}i\,Q^{-1/2} + g'' + g'^2 \pm 2i\,Q^{1/2} g' \sim 0$ Assume $g'' \ll g'^2$

$g'^2 \pm 2i\,Q^{1/2} g' \pm \frac{1}{2} i\, Q^{-1/2} \sim 0$

~~$g'^2 \pm \frac{1}{2}i\, Q^{-1/2} \sim 0 \to g' \sim \sqrt{\mp\frac{1}{2}i}\; Q^{-1/4}$~~

Since $g' \ll f'$, $g'^2 \ll \pm 2i\,Q^{1/2} g'$

$\pm 2i\,Q^{1/2} g' \pm \frac{1}{2} i\, Q^{-1/2} \sim 0$

$g' \sim -\frac{1}{4} Q^{-1/4}$, $g'' \sim \left(\frac{1}{4}\right)^2 Q^{-5/4} Q'$

$\frac{(g'')}{(g')^2} \sim \frac{Q^{-5/4} Q'}{Q^{-2/4}} \sim Q^{-3/4}(x) Q' \ll 1$

Solutions clearly fail for $Q(x) = 0$, $Q'(x) = 0$.

$f' \sim \pm i\sqrt{Q} \to f \sim \pm i\int Q^{1/2}\,dx'$

$g' \sim -\frac{1}{4} Q^{-1/4} \qquad g \sim -\frac{1}{4}\int Q^{-1/4}\,dx'$

$\psi \sim e^{f+g} \sim e^{\pm i \int Q^{1/2} dx'}\, e^{-\frac{1}{4}\int Q^{-1/4}\,dx'}$