Some of the frame transformations are confusing.

2011 Day 1 Q2 (Waves)

a. In lab frame, $\phi \sim e^{i(kx - \omega t)}$, so ϕ is static in frame moving at phase velocity $v_p = \omega/k$.

$$\tfrac{1}{2} m w^2 = \tfrac{1}{2} m w_0^2 + e\phi$$

$$w = \sqrt{w_0^2 + \frac{2e\phi}{m}}$$

K

$K' \longrightarrow \omega/k$

so $v' = v - \frac{\omega}{k}$

$w = v - \frac{\omega}{k}$

$w = w_0 - \frac{\omega}{k}$

b. $$|w|\, dn = |w_0|\, n_0 f_w(w_0)\, dw_0$$

$$n = n_0 \int \frac{|w_0|}{|w|} f_w(w_0)\, dw_0$$

$$= n_0 \int \left(1 + \frac{2e\phi}{m w_0^2}\right)^{-1/2} f_w(w_0)\, dw_0$$

$$n \approx n_0 \int \left(1 - \frac{e\phi}{m w_0^2}\right) f_w(w_0)\, dw_0$$

Poisson equation: $-\nabla^2 \phi = 4\pi \sum_s q n = 4\pi e(n_i - n_e)$

Assume $n_i = n_0$, FT $\phi \rightarrow k^2 \phi = 4\pi e n_0 \left(1 - \int \left(1 - \frac{e\phi}{m w_0^2}\right) f_w(w_0)\, dw_0\right)$

$$\phi - \frac{4\pi e n_0}{k^2}\left[1 - \int (\quad)\, dw_0\right] = 0$$

$$\phi - \frac{4\pi e n_0}{k^2}\left[1 - \int f_w(w_0)\, dw_0 + \int \frac{e\phi}{m w_0^2} f_w(w_0)\, dw_0\right] = 0$$

$$1 - \frac{\omega_p^2}{k^2} \int \frac{f_w(w_0)}{w_0^2}\, dw_0 = 0 \qquad \text{where } w_0 = v - \frac{\omega}{k}$$

$$1 - \frac{\omega_p^2}{k^2} \int \frac{f_0(v)}{(v - \omega/k)^2}\, dv = 0$$

c. $$\frac{1}{(v - \omega/k)^2} = \frac{1}{(\omega/k)^2} \frac{1}{(1 - kv/\omega)^2} \approx \frac{1}{(\omega/k)^2}\left(1 + \frac{2kv}{\omega} + 3\left(\frac{kv}{\omega}\right)^2\right)$$

$$\rightarrow \cancel{1 - \frac{\omega_p^2}{k^2}\int \frac{f_0(v)}{(v - \omega/k)^2}\, dv \approx 1 - \frac{\omega_p^2}{k^2}\left(\frac{k}{\omega}\right)^2\left[\int f_0(v)\, dv + \frac{2k}{\omega}\int v f_0(v)\, dv\right] = 0}$$

$$0 \approx 1 - \frac{\omega_p^2}{\omega^2}\left[\int f_0(v)\, dv + \frac{2k}{\omega}\int v f_0(v)\, dv + \left(\frac{3k}{\omega}\right)^2 \int v^2 f_0(v)\, dv\right]$$

Assume Maxwellian $f_0 = \frac{1}{\sqrt{2\pi}\, v_t} e^{-v^2/v_t^2}$

$$\rightarrow \omega^2 = \omega_p^2\left(1 + 3\frac{k^2 v_t^2}{\omega^2}\right)$$