2011 Day 1 5B (Asymptotics)

a. $y'' - xy' - y = 0$

$x \to \infty$, $t = -\frac{1}{x}$ $\frac{dt}{dx} = -\frac{1}{x^2} = -t^2$, $\frac{d^2t}{dx^2} = \frac{2}{x^3} = 2t^3$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$, $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2 + \frac{dy}{dt}\frac{d^2t}{dx^2}$

$t^4\ddot{y} + 2t^3\dot{y} - t\dot{y} - y = 0$ $t \to 0$ irregular.

$y \sim e^{S(x)}$, $S \sim x^p$

$S'' + S'^2 - xS' - 1 \sim 0$ $S'' \ll S'^2$

$S'^2 - xS' - 1 \sim 0$

1. $S' \sim \pm 1$ weak balance

2. $S'(S' - x) \sim 0$, $S' \sim x$ ✓

3. $S' \sim \frac{-1}{x}$ also works ~~but $S'' \ll S'^2$ (check coefficient)~~

To next order,

balanced

$\cancel{f''} + g'' + \cancel{f'^2} + g'^2 + 2f'g' - \cancel{xf'} - xg' - \cancel{1} \sim 0$ $g'' \ll g'^2$

$g'^2 + x g' \sim 0$

$g'(g' + x) \sim 0 \to g' \sim -x$ not $\ll f' \to g' \sim 0$ only.

~~try $g' \sim Ax^{-1} \to g'' + g'^2 + xg' \sim 0$~~

~~$-Ax^{-2} + A^2x^{-1} +$~~

One solution: $e^{\frac{1}{2}x^2}e^{C}$

Second solution: ~~$f' \sim Ax^{-1}$, $f'' + f'^2 - xf' - 1 \sim 0$~~ $e^{\log x^{-1}} \sim x^{-1}$

b. $x \to 0$ ordinary point : $y = \sum_0 a_n x^n$

$\sum n(n-1)a_n x^{n-2} - \sum n a_n x^n - \sum a_n x^n$ let $j = n-2 \Leftrightarrow n = j+2$

$\sum_{j=-2}(j+1)(j+2)a_{j+2}x^j = \sum_{j\geq 0}(j+1)(j+2)a_{j+2}x^j$

$\to \sum_{n\geq 0}\left[(n+1)(n+2)a_{n+2} - (n+1)a_n\right]x^n = 0$

$$a_{n+2} = \frac{a_n}{n+2} \quad n \geq 0$$

$y_1 = \sum_{n\geq 0} a_{2n}x^{2n}$ $a_{2n} = \frac{a_0}{2^n\Gamma(n+1)}$ $n \geq 1$

$y_2 = \sum_{n=1} a_{2n+1}x^{2n+1}$ $a_{2n+1} = \frac{\Gamma(1/2)}{2^n\Gamma(n+3/2)}a_0$ $n \geq 1$

e. $y(x) = \int_c e^{xt} f(t)\, dt$ s.t. $y(x \to +\infty) \to 0$

$\int_c (t^2 - xt - 1) e^{xt} f(t)\, dt$

$u = t f(t)$ $dv = x e^{xt}$

$du = f(t) + t f'(t)$ $v = e^{xt}$

$- t f(t) e^{xt} \Big|_c + \int_c \left[t^2 f(t) - \left(f(t) + t f'(t) \right) - f(t) \right] e^{xt}\, dt = 0$

$\int_c \left[t^2 f(t) - t f'(t) \right] e^{xt}\, dt = 0$

$\to t f(t) = f'(t) \to f(t) = A e^{\frac{1}{2}t^2}$

$\to t e^{\frac{1}{2}t^2 + xt} \Big|_c \overset{=}{\cancel{\ }} 0$ exactly at $t = 0$

or as