2012 Day 2, Question 3 (Asymptotics)

a. $x \to \infty$

$xy'' + y' - xy = 0$ $\qquad t = \frac{1}{x}$, $dt = -\frac{1}{x^2}dx$, $\frac{d^2t}{dx^2} = \frac{2}{x^3}$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$, $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2 + \frac{dy}{dt}\frac{d^2t}{dx^2}$

$\frac{1}{t}\left(t^4\ddot{y} + 2t^3\dot{y}\right) - 2t^2\dot{y} - \frac{1}{t}y = 0$

$\ddot{y} + \frac{2}{t}\dot{y} - \frac{2}{t}\dot{y} - \frac{1}{t^4}y = 0$ $\qquad$ irregular singular $t \to 0$

Try $y \sim e^{S(x)}$, $\cancel{x}$ $S(x) \sim x^p$

$xS'' + xS'^2 + S' - x \sim 0$

$xS'^2 + S' - x \sim 0$ $\qquad$ (assume $S'' \ll S'^2$)

1. $S'(S'x + 1) \sim 0 \to S' \sim -\frac{1}{x}$ weak balance
2. $x(S'^2 - 1) \sim 0$, $S' \sim \pm 1$ ✓ $\to S \sim \pm x$
3. $S' \sim x$, weak balance.

Now suppose $f' + g'$, $f' \sim \pm 1$, $g' \ll f'$.

$x(\underbrace{\cancel{f''}}_{0} + g'') + \underbrace{\cancel{xf'^2}}_{\text{balance}} + xg'^2 + 2xf'g' + f' + \underbrace{\cancel{g'}}_{\ll f'} \cancel{- x} \sim 0$ $\qquad$ balanced

$xg'' + xg'^2 \pm 2xg' \pm 1 \sim 0$ $\qquad (g'' \ll g'^2)$

$xg'^2 \pm 2xg' \pm 1 \sim 0$

1. $xg'(g' \pm 2) \sim 0$, $g' \sim \mp 2$ not $\ll f'$
2. $xg'^2 \pm 1 \sim 0$ $\quad g' \sim \sqrt{\mp 1/x}$ & weak balance
3. $2xg' + 1 \sim 0$, $g' \sim \frac{-1}{2x}$ ✓ $\to g \sim -\frac{1}{2}\log x$

$y \sim \frac{1}{\sqrt{x}} e^{\pm x}$

b. $x \to 0$ regular singular point. Frobenius solution. $y = \sum_0 a_n x^{n+r}$

$\sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-1} + \sum a_n (n+r) x^{n+r-1} - \sum a_n x^{n+r+1}$

$n-1 = j+1$, $n = j+2$, $j = n-2$

$\sum_{j=-2} a_{j+2}(j+r+2)(j+r+1)x^{j+r+1} + a_{j+2}(j+r+2)x^{j+r+1} - \sum_{j=0} a_j x^{j+r+1}$

$j=-2$ $\qquad a_0\left[r(r-1) + (r-1)\right]x^{r-1} = 0$ $\qquad (r+1)(r-1) = 0$

$j=-1$ $\qquad a_1\left[r(r+1) + (r+1)\right]x^{r} = 0$ $\qquad (r+1)^2 = 0$

Recursion relation $\quad a_{n+2}(n+r+2)^2 = a_n \qquad n \geq 0$

$r = -1$, $a_0 = 0$ $\qquad a_{n+2} = \frac{a_n}{(n+1)^2} \qquad n \geq 1$

$r = 1$, $a_1 = 0$ $\qquad a_{n+2} = \frac{a_n}{(n+3)^2} \qquad n \geq 0$

c. $y(z) = \int_c e^{zt} f(t)\,dt$

$$\int_c \left(z t^2 f(t) + t f(t) - z f(t) \right) e^{zt}\,dt = 0$$

$\cancel{u = t^2 f(t) \qquad dv = z e^{zt}}$ $\qquad u = (t^2-1) f(t) \qquad dv = z e^{zt}$

$\cancel{du = 2tf + t^2 f' \qquad v = \tfrac{1}{z} e^{zt}}$ $\qquad du = 2t f(t) + (t^2-1) f' \qquad v = e^{zt}$

$$(t^2-1) f(t) e^{zt} \Big|_c - \int_c \left[t f(t) + (1-t^2) f'(t) \right] e^{zt}\,dt = 0$$

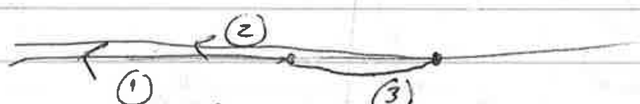
$\rightarrow \quad \int \frac{df}{f} = \int \frac{t\,dt}{t^2-1} \qquad u = t^2-1, \quad du = 2t\,dt$

$\log f = \log\left(\sqrt{t^2-1}\right) + c$

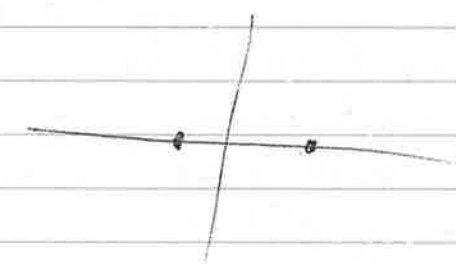
$f = A\sqrt{t^2-1}$

$A(t^2-1)^{3/2} e^{zt} \Big|_c = 0$ vanishes at $t \to \pm 1$, $t \to -\infty$
(for real z)

Any two of these 3 contours may be chosen to construct 2 independent solutions.



$$y(z) = A \int_c e^{zt} \sqrt{t^2-1}\,dt$$



b) $xy'' + y' - xy = 0$ (2012, Day 2, Question 3, page 2)

(mostly done)

$x \to 0$ regular, $y = \sum_0 a_n x^{n+r}$

$$\sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0} a_n (n+r) x^{n+r-1} - \sum_{n=0} a_n \cancel{(n+r)} x^{n+r+1}$$

$$\sum_{j=-2} a_{j+2} (j+r+2)(j+r+1) x^{j+r+1}$$

$n-1 = j+1$

~~$j = n+2$, $n = j+2$~~

$$+ \sum_{j=-2} a_{j+2} (j+r+2) x^{j+r+1}$$

$n = j+2$, $j = n-2$

$$- \sum_{j=0} a_j x^{j+r+1}$$

$j=-2$ $\quad a_0 \left(r(r-1) + r \right) = 0$

$r^2 a_0 = 0$

$j=-1$ $\quad a_1 \left(r(r+1) + (r+1) \right) = 0$

$a_1 (r+1)^2 = 0$

~~straightforward path to two independent solutions via odd and even parity~~.

1. $r = 0 \to a_1 = 0$

   even powers

2. $r = -1$, $a_0 = 0$

   $\sum_{n=0} a_n x^{n-1}$ $\quad j = n-1$

   $\sum_{j=-1} a_{j+1} x^j$

   $\sum_{j=0} a_{j+1} x^j$

   $a_1 x^0 + a_3 x^2 + \dots$

these give the same series.

use the

$y_1 \lg x + \sum b_n x^{n+r}$

$b_n = \left. \frac{\partial a_n}{\partial r} \right|_{r=s}$

2012 Day 2 Question 3 (Asymptotics)

(mostly done)

c) $x y'' + y' - xy = 0$

$y(x) = \int_C f(t) e^{xt}\,dt$

$\int x t^2 f(t) e^{xt} + t f(t) e^{xt} - x f(t) e^{xt}$

$u = t^2 f$ $\quad dv = x e^{xt}$ $\qquad u = f$ $\quad dv = x e^{xt}$

$du = 2tf + t^2 f'$ $\quad v = e^{xt}$ $\qquad du = f'$ $\quad v = e^{xt}$

$t^2 f e^{xt}\Big| - \int (2tf + t^2 f') e^{xt}$ $\qquad f e^{xt}\Big| - \int f' e^{xt}$

$\int \cancel{-2t^2 f} \left(-(t^2-1) f' - tf\right) e^{xt} + (t^2 - 1) f e^{xt}\Big|_C = 0$

$(t^2-1) f' + tf = 0$

$f' + \left(\frac{t}{t^2-1}\right) f = 0$

$\int \frac{df}{f} = -\int \frac{t}{t^2-1}\,dt$ $\qquad u = t^2 - 1$, $du = 2t\,dt$

$\log f = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\log(t^2-1) + c$

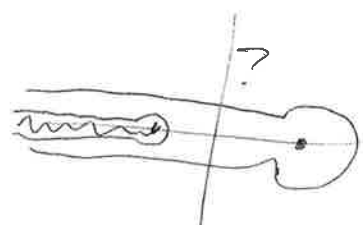
$f(t) = \frac{A}{\sqrt{t^2-1}}$

$A\frac{(t^2-1)}{\sqrt{t^2-1}} e^{xt}\Big|_C = 0$

$A\sqrt{t^2-1}\, e^{xt}\Big|_C = 0$

0 at $t = \pm 1$, $t \to -\infty$

and these are branch points???



- decide how branch cuts work
- decide how to do the ~~asymptotics~~ integral evaluation (necessary?)

a) $xy'' + y' - xy = 0$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$ $\quad x = \frac{1}{t}$ $\quad dx = -\frac{1}{t^2}\,dt$, $\quad \frac{dt}{dx} = -\frac{1}{x^2} = -t^2$, $\quad \frac{d^2t}{dx^2} = \frac{2}{x^3} = 2t^3$

$\frac{1}{t}\left(t^4 \ddot{y} + 2t^3 \dot{y}\right) - t^2 \dot{y} - \frac{1}{t} y = 0$

$\ddot{y} + \frac{2}{t}\dot{y} - \frac{\dot{y}}{t} - \frac{1}{t^4} y = 0 \to$ irregular $x \to \infty$.

$xS'' + x(S')^2 + S' - x \sim 0$ $\longrightarrow$ $\cancel{xf''}^{\,0} + xg'' + \cancel{xf'^2} + xg'^2 + 2xf'g' + f' + g' - \cancel{x} \sim 0$ $\quad \ll f'$ balanced

$x(S')^2 + S' - x \sim 0$ $\qquad xg'^2 \pm 2xg' \pm 2 + \cancel{g'} \sim 0$

1. $S'(xS'+1) \sim 0$ ~~misses~~ weak balance
2. $x(S'^2 - 1) \sim 0$ $\quad S' \sim \pm 1$ ✓
3. $S' \sim x$ weak.

1. $xg'(g' \pm 1) \to g' \sim \mp 1$ not $\ll f'$
2. $xg'^2 \pm 2 \sim 0$, $g' \sim \sqrt{\mp 2x^{-1}}$ weak
3. $\pm 2(xg' + 1) \sim 0$, $g' \sim -\frac{1}{x}$

violates $g'' \ll g'$ but $-\frac{1}{x}$ balances those terms perfectly as well, and they are less important.

$x \to \infty$ $\quad \left(\sim \frac{1}{x} e^{\pm iX}\right)$

$$\chi_s = \frac{2\omega_p^2}{k^2 v_t^2}\left(1 + \frac{\omega}{k v_t} Z(\xi)\right)$$

assume $\frac{\omega_p^2}{\omega^2} \gg 1$

$T_e \gg T_i$

$\xi = \frac{\omega}{k v_T \sqrt{2}}$ hot $\rightarrow$ v_T large $\rightarrow$ ~~$\xi \gg 1$~~ $\xi \ll 1$

warm $\rightarrow$ v_T small, nonzero $\rightarrow$ $\xi \gg 1$

$$\chi_e(\xi \ll 1) = \frac{2\omega_{pe}^2}{k^2 v_e^2}\left(1 + \frac{\omega}{k v_e}\left[i\pi e^{-\xi^2} - 2\xi\left(1 - \frac{2\xi^2}{3}\right)\right]\right)$$