2012 Day 2 Question 4 (MHD)

1. $\nabla \cdot V = 0$ (incompressible)

$\nabla^2 \phi = 0$

$z > 0$ $\quad \nabla^2 \left(\phi_1(z) e^{i(kx - \omega t)} \right) = -k^2 \phi_1(z) + \phi_1''(z) = 0$

$\phi_1''(z) = k^2 \phi_1(z) \rightarrow \phi_1(z) = A e^{-kz}$ (choose $-kz$ to prevent blowup)

$z < 0$ Same calculation with $\phi_2(z) = B e^{kz}$

2. $V_z = \frac{d\xi}{dt}$

$z > 0$ $\quad (\partial_t + U_0 \partial_x) d\, e^{i(kx - \omega t)} = \partial_z A e^{-kz} e^{i(\ )}$

$(-i\omega + ikU_0) d = -kA \cancel{e^{-kz}}$ $\quad z \to 0^+$

$\rightarrow A = i d(\omega - U_0 k)/k$

$z < 0$ $\quad (\partial_t - U_0 \partial_x) d\, e^{(\ )} = \partial_z B e^{kz} e^{(\ )}$

$(-i\omega - ikU_0) d \cancel{e^{(\ )}} = kB$

$\rightarrow B = -id(\omega + U_0 k)/k$

3. $\rho \frac{dV}{dt} = -\nabla P$ $\quad$ Assume P also of the form $P(z) e^{i(kx - \omega t)}$

(not strictly necessary; could be derived from force balance explicitly)

$z > 0$ $\quad \rho_0 (\partial_t + U_0 \partial_x) \partial_x \phi = -\partial_x P$

$\rho_0 (-i\omega + ikU_0)(ik) A e^{-kz} e^{i(\ )} = -ik P$ $\quad z \to 0^+$

$\rho_0 i (-kU_0 + \omega) A = P_1$

$z < 0$ $\quad \rho_0 (\partial_t - U_0 \partial_x) \partial_x \phi = -\partial_x P$

$\rho_0 (-i\omega - ikU_0)(ik) B e^{kz} = -ikP$ $\quad z \to 0^-$

$\rho_0 i (\omega + kU_0) B = P_2$

B.C. $\quad P_1 = P_2$

$\rightarrow -(\omega - U_0 k)^2 \rho_0 d = \rho_0 d (\omega + U_0 k)^2$

Assuming same background density on both sides of the interface,

$\omega = \pm i U_0 k$, $\quad \mathrm{Im}\,\omega > 0$ gives instability here.

Note B_0 is uniform $B_0 \hat{x}$

4. $\eta J = E + v \times B / c$

$\nabla \times \eta J = -\frac{1}{c}\dot{B} + \nabla \times (v \times B)/c = 0$ as $\eta \to 0$ (ideal limit)

$\nabla \times (v \times B) \cong \nabla \times (U \times B_1) + \nabla \times (V_1 \times B_0)$

$\nabla \times (U \times B_1) = U(\nabla \cdot B_1) - B_1(\nabla \cdot U) + (B_1 \cdot \nabla)U - (U \cdot \nabla)B_1$

$= -(U \cdot \nabla)B_1$

$\nabla \times (V \times B_0) = (B_0 \cdot \nabla)V_1$

Let $\vec{B}_1 = \vec{B}(z) e^{i(kx - \omega t)}$, $B_0 = B_0 \hat{U}_0$

$-i\omega B(z) e^{(\ )} = (B_0 \cdot \nabla)V_1 - (U \cdot \nabla)B_1$

$z > 0$ $= B_0 \partial_x^2 \phi - U_0 \partial_x B(z) e^{i(\ )}$

$-i\omega B_x = -Ak^2 B_0 \phi_1 - ikU_0 B_x e^{(\ )}$

$i(kU_0 - \omega)B_x = -Ak^2 B_0 e^{-kz}$

$B_x = \frac{-k^2}{i(kU_0 - \omega)} B_0 A e^{-kz} = B_0 k d$

$z < 0$ $-i\omega B_x = +B_0 \partial_x^2 \phi_2 + U_0 \partial_x B(z)$

$= -k^2 B_0 B e^{kz} + ikU_0 B_x$

$B_x(-i)(\omega + kU_0) = -k^2 B_0 B e^{kz}$

$B_x = \frac{+k^2}{i(\omega + kU_0)} B_0 B e^{kz} = -B_0 k d$

5. $P_1 + \frac{B_{x1} B_0}{8\pi} = P_2 + \frac{B_{x2} B_0}{8\pi}$

$-(\omega - U_0 k)^2 \rho_0 + \frac{B_0^2 k^2}{8\pi} = (\omega + U_0 k)^2 \rho_0 - \frac{B_0^2 k^2}{8\pi}$

$[(\omega + U_0 k)^2 + (\omega - U_0 k)^2] = 2 \cdot \frac{B_0^2 k^2}{\rho_0 8\pi}$

$2(\omega^2 + (U_0 k)^2) = 2 \cdot \frac{B_0^2 k^2}{\rho_0 8\pi}$

$\omega^2 = \frac{B_0^2 k^2}{\rho_0 8\pi} - U_0^2 k^2$

$= k^2 \left(\frac{V_A^2}{2} - U_0^2 \right)$

$V_A^2 > 2U_0^2$ for stability ($\operatorname{Im}\omega = 0$)