2012 Day 2 Question 5 (Computation)

1. PDE $\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0$

$$\frac{U_{j+\frac{1}{2}}^{n+1} - U_{j+\frac{1}{2}}^{n}}{\delta t} = -a\left(\frac{U_{j+1}^{n+1/2} - U_{j}^{n+1/2}}{\delta x}\right)$$

~~consider $f(x+h/m) \approx f(x) + \frac{h}{m}f'(x) + \frac{1}{2}(h/m)^2 f''(x) + O((h/m)^3)$~~

~~$f(x-h/m) \approx f(x) - \frac{h}{m}f'(x) + \frac{1}{2}(h/m)^2 f''(x) - O((h/m)^3)$~~

~~$\frac{f(x+h/2) - f(x-h/2)}{h} = f'(x) + O((h/m)^3)$~~

In Δt and Δx, $f(x+h) \approx f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \cdots$

$\rightarrow \frac{f(x+h) - f(x)}{h} = f'(x) + O(h^2)$ so leading order correction is $\Delta x^2, \Delta t^2$

2. FT so error scales like $e^{bt}e^{ikx}$

$$e^{bt}e^{ikx}e^{ik\Delta x/2}\left(e^{b\Delta t} - 1\right) = -a\frac{\Delta t}{\Delta x}e^{ikx}e^{bt}\left(e^{ik\Delta x} - 1\right)e^{b\Delta t/2}$$

$$\left(e^{b\Delta t/2} - e^{-b\Delta t/2}\right) = -a\frac{\Delta t}{\Delta x}\left(e^{ik\Delta x/2} - e^{-ik\Delta x/2}\right)$$

$$\sinh\left(\frac{b\Delta t}{2}\right) = -ia\frac{\Delta t}{\Delta x}\sin\left(\frac{k\Delta x}{2}\right).$$

$$\text{gain} = e^{b\Delta t} = \left(\sinh\left(\frac{b\Delta t}{2}\right) + \cosh\left(\frac{b\Delta t}{2}\right)\right)^2$$

$$= \left(1 + 2\sinh\left(\frac{b\Delta t}{2}\right)\right)^2 = \left(1 - 2ia\frac{\Delta t}{\Delta x}\sin\left(\frac{k\Delta x}{2}\right)\right)^2$$

$|\text{gain}| \leq 1$ for stability.

$$\left|1 - 2ia\frac{\Delta t}{\Delta x}\sin(\,) - \left(2a\frac{\Delta t}{\Delta x}\right)^2\sin^2(\,)\right| \leq 1$$

$$\sqrt{\left[1 - \left(\frac{2a\Delta t}{\Delta x}\right)\sin^2\right]^2 + \left(4a\frac{\Delta t}{\Delta x}\right)^2\sin^2} \leq 1$$

$$\sqrt{1 + 8\left(a\frac{\Delta t}{\Delta x}\right)^2\sin^2(\,) + \left(2a\frac{\Delta t}{\Delta x}\right)^4\sin^4(\,)} \leq 1$$

Unless $\Delta t = 0$, this can not be stable...

3. ? Matrix inversion