2012 Day 2 Question 5B

a. Let l be the vibrational distance added to path length.

$\Delta\phi = \Delta\phi^0 + \Delta\phi'$ $\qquad \Delta\phi' = 2\pi \frac{l}{\lambda}$

measured in lab. ; ideal ; perturbation

$\Delta\phi^0 = \int k\lambda n_e \, dl = k\lambda f$

$\lambda_H(\Delta\phi_H - \Delta\phi_H^0) = 2\pi l = \lambda_C(\Delta\phi_C - \Delta\phi_C^0)$

$\lambda_H \Delta\phi_H - \lambda_H^2 k f = \lambda_C \Delta\phi_C - \lambda_C^2 k f$

$\rightarrow f = \int n_e \, dl = \frac{1}{k} \frac{\lambda_C \Delta\phi_C - \lambda_H \Delta\phi_H}{\lambda_C^2 - \lambda_H^2}$

b. $\delta f = \frac{1}{k} \frac{\lambda_C \, \delta(\Delta\phi_C) - \lambda_H \, \delta(\Delta\phi_H)}{\lambda_C^2 - \lambda_H^2}$

Suppose $\delta(\Delta\phi_C) \ll \delta(\Delta\phi_H) \sim \pi$, then

$\delta f \cong \frac{\pi \lambda_H}{k} \frac{1}{\lambda_C^2 - \lambda_H^2}$

c. $\frac{\delta f}{f} \cong \frac{\pi \lambda_H}{\lambda_C^2 - \lambda_H^2} \frac{1}{k \int n_e \, dl}$

To estimate this, we need to find k.

$\Delta\phi = \int \frac{2\pi \, dx}{\lambda} = \int (k - k_0) \, dx = \frac{\omega}{c} \int N - 1 \, dx$

$\omega^2 = \omega_p^2 + k^2 c^2 \rightarrow N^2 = 1 - \frac{\omega_p^2}{\omega^2} \rightarrow N \approx 1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}$ for $\omega \gg \omega_p$

$\Delta\phi = \frac{-1}{2\omega c} \int \frac{4\pi n e^2}{m} dx = \frac{2\pi e^2}{\omega m c} \int n(x) \, dx = \frac{e^2}{mc^2} \lambda \int n(x) \, dx$

So $k = e^2/mc^2 = \frac{(4.8\text{ E}-10 \text{ statC})^2}{(9.1\text{ E}-28\text{ g})(3\text{ E}10 \text{ cm/s})^2} = \frac{24 \text{ E}-20}{27 \text{ E}-8} \approx .9 \text{ E}-12 \approx \text{E}-12$

$\int n_e \, dl \approx (\text{E}14 \text{ cm}^{-3})(\text{E}2 \text{ cm}) \approx \text{E}16 \text{ cm}^{-2}$

And $\frac{\pi \lambda_H}{\lambda_C^2 - \lambda_H^2} \approx \frac{\pi \lambda_H}{\lambda_C^2} = \frac{\pi(.6 \text{ E}-4 \text{ cm})}{(10.6 \text{ E}-4 \text{ cm})^2} \approx \frac{1.8 \text{ E}-4}{110 \text{ E}-8} \approx \text{E}+2$

$\rightarrow \frac{\delta f}{f} \approx \frac{(\text{E}2 \text{ cm}^{-1})}{(\text{E}-12 \text{ cm})(\text{E}16 \text{ cm}^{-2})} = 1\%.$

d. $\Delta\phi_H^0 \approx k\lambda_H f = (\text{E}-12 \text{ cm})(.6 \text{ E}-4 \text{ cm})(\text{E}16 \text{ cm}^{-2}) \approx .6 \approx \pi/5$

$\Delta\phi_C^0 \approx (\text{ " " })(10.6 \text{ E}-4 \text{ cm})(\text{ " " }) \approx 10.6 \approx 3\pi$