2012 Day 1 Question 2 (Waves)

a. $\frac{\partial f}{\partial t} + v \cdot \nabla f + F \cdot \nabla_p f = 0$

$\frac{\partial f_1}{\partial t} + v \cdot \nabla f_1 + \tilde{F} \cdot \nabla_p f_0 = 0 \qquad \tilde{F} = q E = -q \nabla \phi$
$\rightarrow -ikq\phi$

$(-i\omega + i k \cdot v) f_1 = + q\, i \phi\, k \cdot \nabla_p f_0$

$f_1 = \frac{q\, \phi\, k \cdot \nabla_p f_0}{k \cdot v - \omega}$

$\nabla \cdot E = -\nabla^2 \phi = -(+ik)^2 \phi = 4\pi \rho = 4\pi \int q f_1\, d^3p$

$\rightarrow \phi = \frac{4\pi q}{k^2} \int f_1\, d^3p = \frac{4\pi q^2}{k^2} \int \frac{\phi\, k \cdot \nabla_p f_0}{k \cdot v - \omega}$

$\epsilon = 0 = 1 - \frac{4\pi q^2}{k^2} \int \frac{k \cdot \nabla_p f_0}{k \cdot v - \omega}\, d^3p .$

$\nabla_p f_0 = \frac{-c}{T} f_0 \rightarrow \epsilon = 1 + \frac{4\pi q^2 c}{k^2 T} \int \frac{k \cos\theta f_0}{k \cos\theta\, v - \omega}\, p^2 \sin\theta\, d\theta\, dp\, (2\pi)$

Ultrarelativistic, $v \approx c$ (although we integrate over all $p(v)$),

$\epsilon \approx 1 + \frac{4\pi q^2}{k^2 T} \int_0^\infty f_0(p)\, 2\pi p^2 dp \int_0^\pi \frac{\cos\theta}{\cos\theta - U}\, d\theta \sin\theta \quad \left(U = \frac{\omega}{kc} \right)$

b. $\epsilon = 1 + \frac{2\pi q^2 n_0}{k^2 T} \int_0^\pi \frac{\cos\theta}{\cos\theta - U} \sin\theta\, d\theta \qquad$ let $s = \cos\theta$

$= 1 + \frac{1}{2 k^2 \lambda_D^2} \int_{-1}^{1} \frac{s}{s - U}\, ds$

For $U \in (-1, 1)$ the denominator goes to zero, so the integral diverges. We are then compelled to interpret the integral with the Landau contour, namely bending the integration contour below the singularity at $\mathrm{Im}\,\omega = 0$. Technically our Laplace Transform above was for $\mathrm{Im}\,\omega > 0$, so to extend to $\mathrm{Im}\,\omega \leq 0$ as well, we need to use the Landau Contour. $\rightarrow$

Note $|u|$ since if $u > 0$,

b. $\int_{-1}^{1} \frac{s}{s-u}\, ds = \int \frac{s-u+u}{s-u}\, ds = \int 1 + \frac{u}{s-u}\, ds$

$= 2 + u \int_{-1}^{1} \frac{ds}{s-u}$

For $|u| > 1$, not divergent, can be evaluated as usual.

$= 2 + u \log\left(\left|\frac{1-u}{1+u}\right|\right)$

For $|u| < 1$, must take residue. 1st order pole: $i\pi|u|$

$= 2 + u \log\left|\frac{1-u}{1+u}\right| + i\pi|u|$

$\rightarrow \quad \mathrm{Re}\,\epsilon = 1 + \frac{1}{k^2\lambda_D^2} + \frac{1}{2k^2\lambda_D^2}\, u \log\left|\frac{1-u}{1+u}\right|$

$\mathrm{Im}\,\epsilon = \frac{i\pi|u|}{2k^2\lambda_D^2}$

Note $|u| \rightarrow 1$, $\mathrm{Re}\,\epsilon$ has logarithmic singularity.

c. Some approximations first:

$u \ll 1$ $\begin{cases} \log(1-u) \approx -u - \frac{u^2}{2} = -u\left(1 + \frac{u}{2}\right) \\ \log(1+u) \approx u - u^2/2 = u\left(1 - \frac{u}{2}\right) \end{cases}$

$\rightarrow \quad u \log\left(\frac{1-u}{1+u}\right) \approx -2u^2$

$u \gg 1$ $\quad \log\left|\frac{1-u}{1+u}\right| = \log\left|\frac{\frac{1}{u}-1}{\frac{1}{u}+1}\right| \approx \log\left|\left(1-\frac{1}{u}\right)\left(1-\frac{1}{u}\right)\right| \approx \log\left|1 - \frac{2}{u}\right|$

$\log\left(1 - \frac{2}{u}\right) \approx -\frac{2}{u} - \frac{1}{2}\left(\frac{2}{u}\right)^2 = -\frac{2}{u}\left(1 + \frac{2}{u}\right)$

$\rightarrow \quad u \log\left|\frac{1-u}{1+u}\right| \approx -2\left(1 + \frac{2}{u}\right)$

so for $u \ll 1$, $\mathrm{Re}\,\epsilon = 1 + \frac{1}{k^2\lambda_D^2} - u^2/k^2\lambda_D^2 \rightarrow 1 + \frac{1}{k^2\lambda_D^2}$ at $u = 0$

$u \gg 1$, $\mathrm{Re}\,\epsilon = 1 + \frac{1}{k^2\lambda_D^2} - \frac{1}{k^2\lambda_D^2}\left(1 + \frac{2}{u}\right) \rightarrow 1$ at $u \rightarrow \infty$.

And there is a logarithmic singularity at $u \rightarrow 1$.