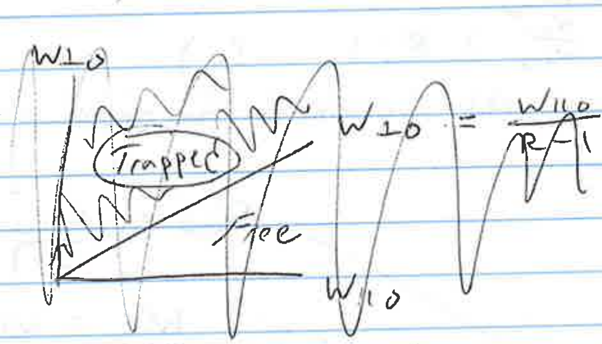
2012 Day 1 Question 3 (GPP)

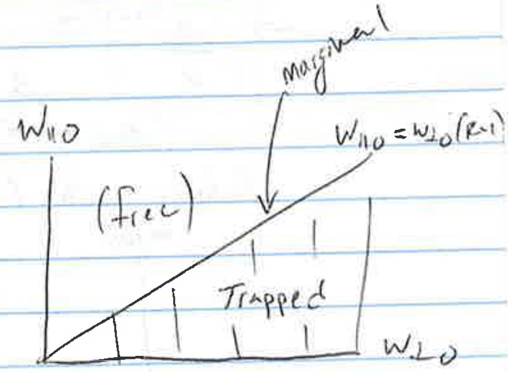
a. $\mu B_{max} > W_0$ trapping

$W_{\perp 0} \frac{B_{max}}{B_0} > W_{\perp 0} + W_{\parallel 0}$

$W_{\perp 0} \left(R - 1 \right) > W_{\parallel 0}$

b.





c. Turning points defined by $\frac{W_{\parallel 0}}{W_{\perp 0}} = \frac{B(z_T)}{B_0} - 1$

$\rightarrow \frac{W_{\parallel 0}}{W_{\perp 0}} = \left(\frac{z}{L}\right)^n$

d. μ is an exact invariant.

$\mu = \frac{W_{\perp 0}}{B_0} = \frac{W_{\perp 0}'}{\beta B_0} \rightarrow W_{\perp 0}' = \beta W_{\perp 0}$

e. $J = \oint p\, dq$ is an adiabatic invariant. $p = m v_\parallel$, $dq = dz$

$\underbrace{\tfrac{1}{2} m v_\parallel^2(z)}_{W_\parallel(z)} + \mu B(z) = W_{\parallel 0} + W_{\perp 0}$

$W_\parallel(z) + W_{\perp 0} \frac{B(z)}{B_0} = W_{\parallel 0} + W_{\perp 0}$

$W_\parallel(z) = W_{\parallel 0} + W_{\perp 0}\left(1 - \left(1 + \frac{z}{L}\right)^n\right)$

$v_\parallel(z) = \sqrt{\frac{2}{m}} \sqrt{W_{\parallel 0}} \sqrt{1 - \frac{W_{\perp 0}}{W_{\parallel 0}} \left(\frac{z}{L}\right)^n}$

$J = \sqrt{2m} \sqrt{W_{\parallel 0}} \int_{z_-}^{z_+} \sqrt{1 - \frac{W_{\perp 0}}{W_{\parallel 0}} \left(\frac{z}{L}\right)^n}\, dz.$

Let $u = \frac{W_{\perp 0}}{W_{\parallel 0}} \left(\frac{z}{L}\right)^n$, $du = \frac{W_{\perp 0}}{W_{\parallel 0}} \frac{n}{L^n} z^{n-1} dz = \frac{n}{L} \left(\frac{W_{\parallel 0}}{W_{\perp 0}}\right)^{-1/n} u^{\frac{n-1}{n}} dz$

$J = \sqrt{2m}\, L\, W_{\parallel 0}^{1/2} \left(\frac{W_{\parallel 0}}{W_{\perp 0}}\right)^{1/n} \int_{-1}^{1} u^{\frac{1-n}{n}} \sqrt{1-u}\, du$

$\rightarrow L \left(\frac{W_{\parallel 0}^{\frac{2+n}{2}}}{W_{\perp 0}}\right)^{1/n} = \alpha L \left(\frac{W_{\parallel 0}'^{\frac{2+n}{2}}}{\beta W_{\perp 0}}\right)^{1/n}$

$\left(\frac{\beta^{1/n}}{\alpha}\right)^{\frac{2n}{2+n}} W_{\parallel 0} = W_{\parallel 0}'$

f. Initially trapped: $\frac{W_{\parallel 0}}{W_{\perp 0}} < R-1$ (1)

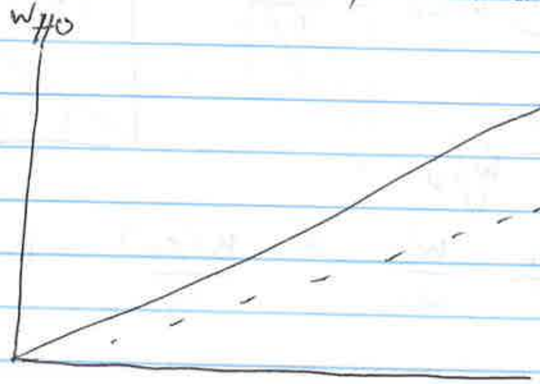
Finally ~~trapped~~ free: $\frac{W_{\parallel 0}'}{W_{\perp 0}'} > R-1$

$$\rightarrow \left(\frac{\beta^{1/n}}{\alpha}\right)^{\frac{2n}{2+n}} \frac{1}{\beta}\,\frac{W_{\parallel 0}}{W_{\perp 0}} > R-1$$

$$(\beta\alpha^2)^{\frac{-n}{2+n}}\,\frac{W_{\parallel 0}}{W_{\perp 0}} > R-1 \qquad (2)$$

Particles with (1) and (2) true transition from trapped → free.

g.



$W_{\parallel 0} = W_{\perp 0}(R-1)$

$W_{\parallel 0} = W_{\perp 0}(R-1)(\beta\alpha^2)^{\frac{n}{n+2}}$

Below —— : initially trapped

Below - - - : finally trapped

h. $(\beta\alpha^2)^{\frac{n}{n+2}} < 1$ so that particles transition from initially trapped to finally free.