(when to use $\int_L$ instead of $\int_{-\infty}^{\infty}$?)

2013 Day 2 Question 2 (Waves)

a. $f = n_0 f_0(v) + f_1(x, v, t)$

Assume $B_0 = 0$, $E_0 = 0$ but $\tilde{E}, \tilde{B} \neq 0$.

Vlasov Equation $\frac{\partial f}{\partial t} + v \cdot \nabla f + \left(\frac{q}{m}\right)\left(E + \frac{v \times B}{c}\right) \cdot \nabla_v f = 0$

$$\frac{\partial f_1}{\partial t} + v \cdot \nabla f_1 = -\left(\frac{q}{m}\right)\left(\tilde{E} + \frac{v \times \tilde{B}}{c}\right) \cdot \nabla_v f_0 n_0$$

(note $(\tilde{E} + v \times \tilde{B}) \cdot \nabla_v f_1$ quadratic in small quantities)

Fourier-Laplace Transform: $f_1(x, v, t) = \int e^{-i(\omega t - kx)} f_1(k, v, \omega)$

so $\nabla \to ik$, $\partial_t \to -i\omega$.

Also FL transform fields s.t. $\nabla \times E = \frac{-1}{c}\frac{\partial B}{\partial t} \to ik \times E = \frac{i\omega}{c} B \to B = \frac{c}{\omega} k \times E$.

$$\cancel{-i\omega + i}\; i((k \cdot v) - \omega) f_1 = -\left(\frac{q}{m}\right)\left(\tilde{E} + \frac{1}{\omega} v \times (k \times E)\right) \cdot \nabla_v f_0 n_0$$

Rewrite $\left[E + \frac{1}{\omega}(v \times (k \times E))\right]_\ell = \left[E + \frac{k}{\omega}(E \cdot v) - \frac{E(k \cdot v)}{\omega}\right]_\ell$

$= E_\ell \left(1 - \frac{k \cdot v}{\omega}\right) + \frac{k_\ell E_j V_j}{\omega} = \left[\delta_{\ell j}\left(1 - \frac{k \cdot v}{\omega}\right) + \frac{k_\ell V_j}{\omega}\right] E_j$ (implicit sum over j)

$$\to f_1 = \frac{-i\left(\frac{q}{m}\right) n_0}{\omega - k \cdot v}\left[\delta_{\ell j}\left(1 - \frac{k \cdot v}{\omega}\right) + \frac{k_\ell V_j}{\omega}\right] \frac{\partial f_0}{\partial v_\ell} E_j \quad (\text{sums over } j, \ell = x, y, z)$$

$$J_{induced} = \int q \vec{v} f_1 = \frac{-i\omega_p^2}{4\pi\omega} \int \frac{\vec{v}}{1 - \frac{k \cdot v}{\omega}}\left[\delta_{\ell j}\left(1 - \frac{k \cdot v}{\omega}\right) + \frac{k_\ell V_j}{\omega}\right] \frac{\partial f_0}{\partial v_\ell} E_j$$

Since we have transformed to k-ω space, we know $J(k, \omega) = \sigma(k, \omega) E(k, \omega)$

$$\to \sigma_{ij} = \frac{-i\omega_p^2}{4\pi\omega} \int \frac{V_i}{1 - \frac{k \cdot v}{\omega}}\left[\delta_{\ell j}\left(1 - \frac{k \cdot v}{\omega}\right) + \frac{k_\ell V_j}{\omega}\right] \frac{\partial f_0}{\partial v_\ell} d^3v$$

$$\chi_{ij} = \frac{4\pi i}{\omega}\sigma_{ij} = \frac{\omega_p^2}{\omega^2} \int \frac{V_i}{1 - \frac{k \cdot v}{\omega}}\left[\delta_{\ell j}\left(1 - \frac{k \cdot v}{\omega}\right) + \frac{k_\ell V_j}{\omega}\right] \frac{\partial f_0}{\partial v_\ell} d^3v$$

Consider the first term $\int \delta_{\ell j} V_i \frac{\partial f_0}{\partial v_\ell} d^3v = \int V_i \frac{\partial f_0}{\partial v_j} d^3v$.

Integrate by parts wrt $\frac{\partial}{\partial v_j}$ $= +V_i f_0 \Big| - \int \frac{\partial V_i}{\partial v_j} f_0 = -\delta_{ij} \int f_0 d^3v = -\delta_{ij}$

$f_0 \to 0$ as $v \to \infty$ so boundary term vanishes. $\frac{\partial V_i}{\partial v_j} = \delta_{ij}$ so this entire term is a δ_{ij}.

$$\chi_{ij} = -\delta_{ij}\frac{\omega_p^2}{\omega^2} + \frac{\omega_p^2}{\omega^3} \int \frac{V_i V_j k_\ell}{1 - \frac{k \cdot v}{\omega}} \frac{\partial f_0}{\partial v_\ell} d^3v \qquad \text{let } k = k\hat{z}$$

$$\epsilon_{ij} = \delta_{ij} + \chi_{ij} = \delta_{ij}\left(1 - \frac{\omega_p^2}{\omega^2}\right) + \frac{\omega_p^2}{\omega^2} \int \frac{V_i V_j k}{\omega - kV_z} \frac{\partial f_0}{\partial v_z} d^3v.$$

b. $f_0(v) = \frac{1}{\pi^{3/2} W_\perp^2 W_\parallel} e^{-\left(\frac{V_x^2+V_y^2}{W_\perp^2} + \frac{V_z^2}{W_\parallel^2}\right)}$ $\qquad W_\parallel^2 = \frac{T_\parallel}{m}$

$$\frac{\partial f_0}{\partial V_z} = -\frac{2V_z}{\pi^{3/2} W_\perp^2 W_\parallel^3} \exp -\left(\frac{V_x^2+V_y^2}{W_\perp^2} + \frac{V_z^2}{W_\parallel^2}\right)$$

$$\epsilon_{ij}(i \neq j) \sim \int \frac{V_i V_j V_z}{\omega - kV_z} \exp -\left(\frac{V_x^2+V_y^2}{W_\perp^2} + \frac{V_z^2}{W_\parallel^2}\right) d^3v$$

exp( ) is an even function of all coordinates.

If $i \neq j$, there will always be an odd powers of ~~exactly~~ at least one of V_x or V_y.

the integration over $\int v_x \, dV_x = \int V_y \, dV_y = 0$ by symmetry,

so $\epsilon_{ij}(i\neq j) = 0$, leaving ϵ diagonal.

$$\epsilon_{xx} = \epsilon_{yy} = 1 - \frac{\omega_p^2}{\omega^2} - \frac{2 \cancel{V_z} k \omega_p^2}{\pi^{3/2} W_\perp^2 W_\parallel^3 \omega} \int \frac{V_z V_x^2}{\omega - kV_z} e^{-\left(\frac{V_x^2+V_y^2}{W_\perp^2} + \frac{V_z^2}{W_\parallel^2}\right)} d^3v$$

$$\int e^{-V_y^2/W_\perp^2} = W_\perp \sqrt{\pi}, \quad \int V_x^2 e^{-V_x^2/W_\perp^2} = \tfrac{1}{2} W_\perp^3 \sqrt{\pi}$$

$$\epsilon_\perp = 1 - \frac{\omega_p^2}{\omega^2} - \frac{k W_\perp^2 \omega_p^2}{\sqrt{\pi} W_\parallel^3 \omega^2} \int \frac{V_z}{\omega - kV_z} e^{-V_z^2/W_\parallel^2} dV_z$$

$$\rightarrow \frac{-k}{\sqrt{\pi}} \int \frac{V_z e^{-V_z^2/W_\parallel^2} dV_z}{\omega - kV_z} = \frac{-k}{\sqrt{\pi}} \int \frac{u W_\parallel e^{-u^2} W_\parallel \, du}{\omega - k u W_\parallel} \qquad \text{let } u = V_z/W_\parallel$$

$\xi = \frac{\omega}{kW_\parallel}$

$$= \frac{W_\parallel}{\sqrt{\pi}} \int \frac{u e^{-u^2}}{u - \xi} du \qquad \text{integrate by parts: } f = \frac{1}{u-\xi} \quad dg = u e^{-u^2}$$
$$df = \frac{-1}{(u-\xi)^2} \quad g = -\tfrac{1}{2} e^{-u^2}$$

$$= -\frac{W_\parallel}{\sqrt{\pi}} \frac{1}{2} \int \frac{e^{-u^2}}{(u-\xi)^2} du \qquad \text{note } Z(\xi) = \frac{1}{\sqrt{\pi}} \int \frac{e^{-u^2}}{u-\xi} du, \quad Z'(\xi) = \frac{+1}{\sqrt{\pi}} \int \frac{e^{-u^2}}{(u-\xi)^2} du$$

$$= -\frac{W_\parallel}{\cancel{\sqrt{\pi}}} \frac{1}{2} Z'(\xi) \quad \text{and} \quad Z'(\xi) = -2(1 + \xi Z(\xi))$$

$= +W_\parallel (1 + \xi Z(\xi))$. Substitute this into earlier formula for $\epsilon_\perp$,

$$\epsilon_\perp = 1 - \frac{\omega_p^2}{\omega^2} + \frac{W_\perp^2}{W_\parallel^2} \frac{\omega_p^2}{\omega^2} (1 + \xi Z(\xi)) \quad \text{where } \frac{W_\perp^2}{W_\parallel^2} = \frac{T_\perp}{T_\parallel}.$$

Dispersion relation: $N_i N_j - N^2 \delta_{ij} + \epsilon_{ij} = 0$

Transverse waves have $-N^2 + \epsilon_{ij} = 0$

$$\rightarrow \omega^2 - k^2c^2 - \omega_p^2 + \omega_p^2 \left(\frac{T_\perp}{T_\parallel}\right)(1 + \xi Z(\xi)) = 0$$

2013 Day 2 Question 2 (Waves)

c. Hot plasma, $\zeta = \frac{\omega}{kW_{\parallel}} \ll 1$

$$Z(\zeta) \approx i\sqrt{\pi} - 2\zeta\left(1 - \frac{2\zeta^2}{3}\right)$$

small

$$\omega^2 - k^2c^2 - \omega_p^2 + \omega_p^2\left(\frac{T_\perp}{T_\parallel}\right)\left(1 + i\sqrt{\pi}\zeta - \cancel{2\zeta^2}\right) = 0$$

Note by neglecting $O(\zeta^2)$ terms we see $\omega^2 \ll k^2c^2$ since

$$\frac{\omega^2}{k^2c^2} = \frac{\omega^2}{k^2W_\parallel^2}\frac{W_\parallel^2}{c^2} = \zeta^2\frac{W_\parallel^2}{c^2} \ll \zeta^2 \quad \text{(nonrelativistic)}$$

$$\rightarrow \quad \omega_p^2\left(\frac{T_\perp}{T_\parallel}\right)\left(1 + \frac{i\sqrt{\pi}\,\omega}{kW_\parallel}\right) = \omega_p^2 + k^2c^2$$

$$\omega = i\frac{kW_\parallel}{\sqrt{\pi}}\left[1 - \frac{T_\parallel}{T_\perp}\left(1 + \frac{k^2c^2}{\omega_p^2}\right)\right] \qquad (T_\perp > T_\parallel)$$

$\operatorname{Im}\omega > 0$ for small enough k $\rightarrow$ unstable mode.

d. cold plasma, $\zeta = \frac{\omega}{kW_\parallel} \gg 1$

$$Z(\zeta) \approx -\frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2}\right)$$

$$\omega^2 - k^2c^2 - \omega_p^2 + \omega_p^2\left(\frac{T_\perp}{T_\parallel}\right)\left(-\frac{1}{2\zeta^2}\right) = 0$$

$$\omega^4 - \omega^2(\omega_p^2 + k^2c^2) + \omega_p^2k^2W_\parallel^2\left(\frac{T_\perp}{T_\parallel}\right) = 0$$

$$\omega^4 - \omega^2(\omega_p^2 + k^2c^2) + \omega_p^2k^2W_\perp^2 = 0$$

$$\omega^2 = \frac{1}{2}\left((\omega_p^2 + k^2c^2) \pm \sqrt{(\omega_p^2 + k^2c^2)^2 - 4\,\cancel{(\omega_p^2 + k^2c^2)}\,(\omega_p^2k^2W_\perp^2)}\right)$$

$$= \frac{\omega_p^2 + k^2c^2}{2}\left[1 \pm \sqrt{1 - \frac{4\omega_p^2k^2W_\perp^2}{(\omega_p^2 + k^2c^2)^2}}\right] \approx \frac{\omega_p^2 + k^2c^2}{2}\left[1 \pm 1 \mp \frac{2\omega_p^2k^2W_\perp^2}{(\omega_p^2 + k^2c^2)^2}\right]$$

One choice is $\omega^2 = -\frac{\omega_p^2k^2W_\perp^2}{(\omega_p^2 + k^2c^2)}$ $\rightarrow$ there is a solution with $\operatorname{Im}\omega > 0$ (unstable)

The cold plasma approximation was $\zeta = \frac{\omega}{kW_\parallel} \gg 1$

$$\rightarrow \quad \zeta = \frac{i\,\omega_p k W_\perp}{kW_\parallel(\omega_p^2 + k^2c^2)^{1/2}} = i\sqrt{\frac{\omega_p^2}{\omega_p^2 + k^2c^2}}\sqrt{\frac{T_\perp}{T_\parallel}} \gg 1 \rightarrow \frac{T_\perp}{T_\parallel} \gg 1.$$