2013 Day 2 Question 3 (MHD)

1. $\delta(W - \mu K) = 0$

$\int B\cdot\delta B + V\cdot\delta V - \mu\, V\cdot\delta B - \mu\,\delta V\cdot B = 0$

$\int (B - \mu V)\cdot\delta B + (V - \mu B)\cdot\delta V = 0$

Since $\delta V, \delta B$ are arbitrary, in order to vanish the integrand it must be that $B - \mu V = 0$, $V - \mu B = 0 \rightarrow \mu = \pm 1$, $V = \pm B$.

2. $\delta(W - \lambda H) = 0$

$\int V\cdot\delta V + B\cdot\delta B - \frac{1}{2}\lambda A\cdot\delta B - \frac{1}{2}\lambda B\cdot\delta A. = 0$

Note $B\cdot\delta A = (\nabla\times A)\cdot\delta A = A\cdot\nabla\times\delta A - \nabla\cdot(\delta A\times A)$

$= A\cdot\delta B - \nabla\cdot(\delta A\times A)$

Note $\int\nabla\cdot(\delta A\times A) = \oint(\delta A\times A)\cdot da = 0$ if $\delta A\cdot da = 0$ @ bndy.

$\int V\cdot\delta V + (B - \lambda A)\cdot\delta B = 0$

$\rightarrow V = 0$, $B = \lambda A$, so $\nabla\times B = \lambda\nabla\times A = \lambda B$

Recognize this as a force-free equilibrium.

3. $\delta(W - \mu K - \lambda H) = 0$

$\int B\cdot\delta B + V\cdot\delta V - \mu V\cdot\delta B - \mu\delta V\cdot B - \lambda A\cdot\delta B = 0$

~~$\int(B - \mu V)\cdot\delta B$~~

$\int(V - \mu B)\cdot\delta V + (B - \mu V - \lambda A)\cdot\delta B = 0$

$\rightarrow V = \mu B$, $B - \mu^2 B - \lambda A = 0$

$B = \frac{\lambda}{1-\mu^2}A$ (still force free)

4. $\lambda\rightarrow 0 \Rightarrow B - \mu^2 B = 0 \rightarrow \mu = \pm 1$ identical to (a)

$\mu\rightarrow 0 \Rightarrow V = 0$, $B = \lambda A$ identical to part (b)

$\lambda\rightarrow 0$ and $\mu\rightarrow 0 \Rightarrow V = 0$, $B = 0$

(unconstrained minimization of W will just kill off both fields).