2013 Day 1 Question 1 (Waves) [Draft]

$$Z' = -2(1+\zeta Z), \quad \zeta = \frac{\omega}{kV_t\sqrt{2}}$$

hw $\chi = -\frac{1}{2k^2\lambda_D^2} Z' = \frac{1}{k^2\lambda_D^2}\left(1+\zeta Z(\zeta)\right)$

exam $\chi = \frac{2}{k^2\lambda_D^2}\left(1+\zeta\sqrt{2}\, Z(\zeta)\right)$

hot $(\zeta \ll 1)$ $Z(\zeta) = i\sqrt{\pi}e^{-\zeta^2} - 2\zeta\left(1-\frac{2\zeta^2}{3}\right)$

$\simeq i\sqrt{\pi} - 2\zeta\left(1-\frac{2\zeta^2}{3}\right)$

cold $(\zeta \gg 1)$ $Z(\zeta) = i\sqrt{\pi}e^{-\zeta^2} - \zeta^{-1}\left(1+\frac{1}{2}\zeta^{-2}\right)$

$\approx -\zeta^{-1}\left(1+\frac{1}{2}\zeta^{-2}\right)$

(hot) $\chi_e = \frac{2}{k^2\lambda_e^2}\left(1 + i\zeta\sqrt{2\pi} - \cancel{2\zeta^2}\ 2\sqrt{2}\zeta^2\right)$

(cold) $\chi_i = \frac{2}{k^2\lambda_i^2}\left(\text{m+}\ (1-\sqrt{2}) - \frac{\sqrt{2}}{2}\zeta^{-2}\right)$

$\epsilon = 1 + \chi_e + \chi_i$

$$\chi_i = \frac{-k}{k^2\lambda_i^2}\ \frac{2k^2V_t^2}{\omega^2} = -\frac{2\omega_{pi}^2}{\omega^2}$$

$$\chi_e = \frac{2}{k^2\lambda_e^2}\left(1-2\zeta^2 + i\zeta\sqrt{\pi}\right)$$

$$\approx \frac{2}{k^2\lambda_e^2} + \frac{2i\sqrt{\pi}\zeta}{k^2\lambda_e^2}$$

small imaginary part.

$$\epsilon = 1 + \frac{2}{k^2\lambda_e^2} - \frac{2\omega_{pi}^2}{\omega^2} + i\left(\frac{\zeta_e\sqrt{\pi}}{k^2\lambda_e^2}\right)$$

$0 \approx \epsilon_r(\omega_r) + i\omega_i\partial_\omega \epsilon_r(\omega_r) + i\epsilon_i(\omega_r)$

$\longrightarrow \epsilon_r(\omega_r) = 0, \quad \omega_i\partial_\omega\epsilon_r(\omega_r) + \epsilon_i(\omega_r) = 0$

$\epsilon_i(\omega_r) = -\omega_i\frac{\partial\epsilon_r(\omega_r)}{\partial\omega}$

$\hookrightarrow 1 + \frac{2}{k^2\lambda_e^2} - \frac{2\omega_{pi}^2}{\omega_r^2} = 0 \rightarrow \omega_r^2\left(1+\frac{2}{k^2\lambda_e^2}\right) = 2\omega_{pi}^2$

$\omega_r^2 = 2\omega_{pi}^2 / \left(1 + 2/k^2\lambda_e^2\right)$

$= \frac{2k^2C_s^2}{1+k^2\lambda_e^2}$ $\quad C_s^2 = \frac{Z_i T_{0e}}{m_i}$

$\rightarrow \omega_i = \frac{-\epsilon_i(\omega_r)}{\partial_\omega\epsilon_r(\omega_r)}$. $\quad \frac{\partial\epsilon_r}{\partial\omega} = \frac{-4\omega_{pi}^2}{\omega_r^3}$

$$= -\frac{\omega_r\sqrt{\pi}}{k^3\lambda_e^2\sqrt{2}V_t}\left(\frac{\omega_r^3}{-4\omega_{pi}^2}\right)$$

Agreement except for spurious factors ...

must use $k\lambda \ll 1$

$$\epsilon_r(\omega_r, k) = 1 + \frac{1}{k^2 \lambda_{De}^2} - \frac{\omega_{pi}^2}{\omega^2}$$

$$\frac{\partial \epsilon_r(\omega_r, k)}{\partial \omega} = \frac{2\omega_{pi}^2}{\omega^3} = \frac{2V_i^2}{\lambda_i^2 \omega^3} \approx -\omega_r \sqrt{\frac{\pi}{8}\frac{Z_i}{\mu}}$$

$$\epsilon_i(\omega_r, k) = \frac{\xi_e^{(r)} \sqrt{\pi}}{k^2 \lambda_e^2} = \sqrt{\frac{\pi}{2}} \frac{\omega_r}{k^3 V_{te} \lambda_e^2}$$

$$\omega_i = -\frac{\epsilon_i(\omega_r, k)}{\partial_\omega \epsilon_r(\omega_r, k)} = -\sqrt{\frac{\pi}{2}} \frac{\omega_r}{k^3 V_{te} \lambda_e^2} \frac{\omega_r^3}{2\omega_{pi}^2}$$