2013 Day 1 Question 5A (Asymptotics)

a. $xy'' + 2y' + xy = 0$

$x \to 0$ regular singular point. Frobenius solution $y = \sum_0 a_n x^{n+r}$

$\sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-1} + \sum_{n=0} 2a_n (n+r) x^{n+r-1} + \sum_{n=0} a_n x^{n+r+1} = 0$

Let $n-1 = j+1 \leftrightarrow n = j+2,\ j = n-2$

$\sum_{j=-2} a_{j+2} (j+r+2)(j+r+1) x^{j+r+1} + \sum_{j=-2} 2a_{j+2}(j+r+2) x^{j+r+1} + \sum_{j=0} a_j x^{j+r+1} = 0$

$j=-2$ $\quad a_0 x^{r-1} (r(r-1) + 2r) = 0 \to r(r+1) = 0$

$j=-1$ $\quad a_1 x^r (r(r+1) + 2(r+1)) = 0 \to (r+1)(r+2) = 0$

$a_{n+2} \{ (n+r+2)(n+r+1) + 2(n+r+2) \} = -a_n \qquad n \geq 0$

$a_{n+2} = \dfrac{-a_n}{(n+r+2)(n+r+3)} \qquad n \geq 0$

Note the $r=-2$ root will blow up all

b. $xy'' + 2y' + xy = 0$

Let $x = \frac{1}{t}$ to analyze $t \to 0$. $dx = \frac{-1}{t^2}dt$, $\frac{dt}{dx} = -t^2 = -\frac{1}{x^2}$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = -t^2\dot{y}$. $\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2 + \frac{d^2t}{dx^2}\frac{dy}{dt} = t^4\ddot{y} + 2t^3\dot{y}$

$\frac{1}{t}(t^4\ddot{y} + 2t^3\dot{y}) - 2t^2\dot{y} + \frac{1}{t}y = 0$

$\ddot{y} + \frac{2}{t}\dot{y} - \frac{2}{t}\dot{y} + \frac{1}{t^4}y = 0$ irregular singular point.

Try $y \sim e^{S(x)}$, $S \sim x^p$

$xS'' + xS'^2 + 2S' + x \sim 0$ assume $S'' \ll S'^2$

$xS'^2 + 2S' + x \sim 0$

1. $S'(S'x + 2) \sim 0$, $S' \sim \frac{-2}{x}$ does not balance most singular term

2. $x(S'^2 + 1) \sim 0 \to S' \sim \pm i$ ✓

3. $2S' + x \sim 0$, violates $S'' \ll S'^2$ assumption.

Let $S' = f' + g' = \pm ix + g'$ with $g' \ll f'$. balanced

$\cancel{xf''}^{0} + xg'' + \cancel{xf'^2} + xg'^2 + 2xf'g' + 2f' + 2g' + \cancel{x} \sim 0$ ~~assume~~

$xg'^2 \pm 2ixg' \pm 2i + \cancel{2g'}_{\ll f'} \sim 0$

~~1. $xg'(g' \pm 2ix) \sim 0$ violates $f' \gg g'$~~

~~2. $x(g'^2 \pm 2i) \sim 0$ violates $f' \gg g'$~~

~~3. $\pm 2ix/g'$~~

1. $xg'(g' \pm 2ix) \sim 0$ violates $g' \ll f'$

2. $xg'^2 \pm 2i \sim 0 \to g' \sim \sqrt{\mp 2i}\, x^{-1/2}$ does not balance most ~~unstable~~ singular term

3. $\pm 2i(xg' + 1) \sim 0 \to g' \sim x^{-1}$ ✓

So the two asymptotic $x \to \infty$ solutions are $\sim \frac{1}{x}e^{ix}$, $\frac{1}{x}e^{-ix}$.

c,d. Let $y(x) = \int_C e^{xt} f(t)\,dt$ over some contour C.

$xy'' + 2y' + xy = 0$

$\int x t^2 f(t) e^{xt}\,dt + \int 2t e^{xt} f(t)\,dt + \int x e^{xt} f(t)\,dt = 0$. integrate by parts

$u = t^2 f(t) \quad dv = xe^{xt} \qquad u = f \quad dv = xe^{xt}$

$du = 2tf + t^2 f' \quad v = e^{xt} \qquad du = f' \quad v = e^{xt}$

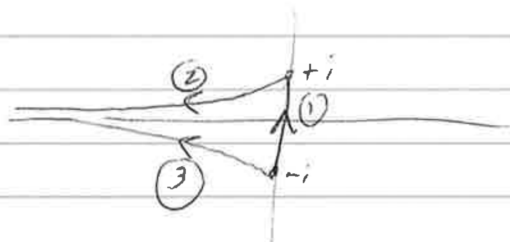
$0 = t^2 f(t) e^{xt}\Big|_C - \int (2tf + t^2 f') e^{xt}\,dt + \int 2t e^{xt} f(t)\,dt + f e^{xt}\Big|_C - \int f' e^{xt}$

$- \int_C (t^2+1) f'(t)\,dt\, e^{xt} + (t^2+1) f(t) e^{xt}\Big|_C = 0$

$\to (t^2+1) f'(t) = 0 \to f(t) = A$, constant function.

Boundary condition becomes $A(t^2+1) e^{xt}\Big|_C = 0$

This will vanish at $t = \pm i$, $t \to -\infty$:



So any of the below contours will work, leaving $y(x) = \int_C A e^{xt}\,dt$

d. Demanding $y(0) = 1$, $y(0) = \int_C A\,dt = AL = 1$ where L is the length of the contour. Only contour ① has finite length, so that must be chosen. Since it has length 2, we can determine $A = \frac{1}{2}$, so $y(x) = \frac{1}{2}\int_{①} e^{xt}\,dt$.