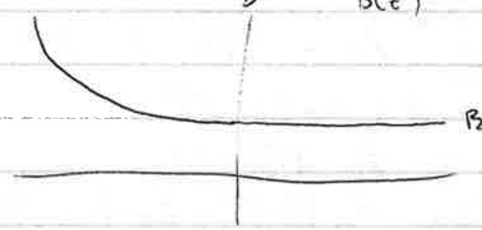
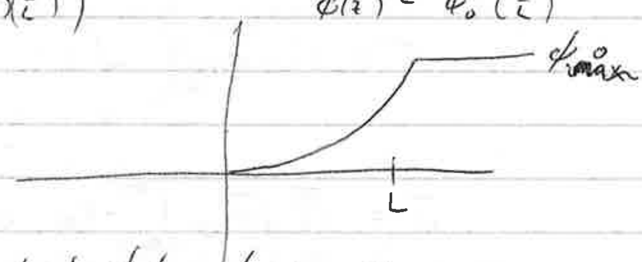
2014 Day 1 Question 1 (GPP)

$B(z) = B_0\left(1 + (R-1)\left(\frac{z}{L}\right)^2\right)$ $\phi(z) = \phi_0\left(\frac{z}{L}\right)^2$

a.





Magnetic trapping as $z \to -\infty$, electrostatic trapping $z \to +\infty$. Electrons would be trapped identically for $z < 0$, but would escape (and in fact be accelerated) as $z > 0$.

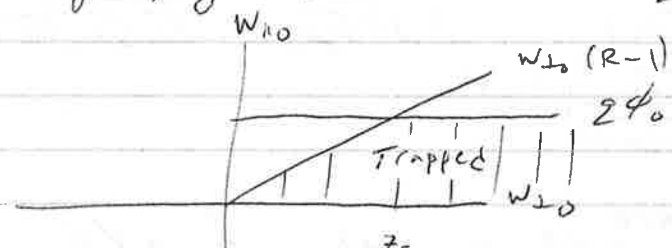
b. For left-going ions, $W_0 < \mu B_{max}$ defines trapping. $\mu = \frac{W_{\perp 0}}{B_0}$

$W_{\parallel 0} + W_{\perp 0} < \frac{B_{max}}{B_0} W_{\perp 0}$

$W_{\parallel 0} < W_{\perp 0}(R-1)$ $R = \frac{B_{max}}{B_0}$.

For right-going ions, $W_{\parallel 0} < q\phi_{max\,0}$ defines trapping.

c.



If ions scattered in angle but not in energy, could leave through either end, but more likely to leave through electrostatic end - it is energy-dependent.

d. $J = \oint p\,dz = m\int_{z_B}^{z_B} V_\parallel(z)\,dz = m\int_{z_B}^{0} + m\int_0^{z_E} V_\parallel(z)\,dz$.

For $z < 0$ portion, $W_\parallel(z) + \mu B(z) = W_{\parallel 0} + W_{\perp 0}$

$W_\parallel(z) = W_{\parallel 0} + W_{\perp 0}\left(1 - \frac{B(z)}{B_0}\right) = W_{\parallel 0}\left(1 - \frac{W_{\perp 0}}{W_{\parallel 0}}(R-1)\left(\frac{z}{L}\right)^2\right)$

$V_\parallel(z) = \sqrt{\frac{2}{m}}\sqrt{W_{\parallel 0}}\sqrt{1 - \frac{W_{\perp 0}}{W_{\parallel 0}}(R-1)\left(\frac{z}{L}\right)^2}$

with turning point given by $V_\parallel(z_B) = 0 \to z_B = -\left(\frac{W_{\perp 0}}{W_{\parallel 0}}\frac{(R-1)}{L^2}\right)^{-1/2} < 0$

For $z > 0$, $W_\parallel(z) = W_{\parallel 0} - q\phi(z)$

$V_\parallel(z) = \sqrt{\frac{2}{m}}\sqrt{W_{\parallel 0}}\sqrt{1 - \frac{q}{W_{\parallel 0}}\phi_0\frac{z^2}{L^2}}$

$V_\parallel(z_E) = 0 \to z_E = \left(\frac{q}{W_{\parallel 0}}\frac{\phi_0}{L^2}\right)^{-1/2}$.

$J \sim \sqrt{W_{\parallel 0}}\int_{-z_B}^{0} dz\sqrt{1 - z^2/z_B^2} + \sqrt{W_{\parallel 0}}\int_0^{z_E} dz\sqrt{1 - z^2/z_E^2}$

$u = z/z_B$, $du = dz/z_B$

$J \sim \sqrt{W_{\parallel 0}}\, z_B\int_1^0 \sqrt{1-u^2}\,du + \sqrt{W_{\parallel 0}}\, z_E\int_0^1 \sqrt{1-u^2}\,du$

$\to \sqrt{W_{\parallel 0}}\,(z_E - z_B) = \sqrt{W_{\parallel 0}}\,(z_E + |z_B|) = \text{constant}.$

d. (continued)

$$J = \sqrt{2m W_{\parallel 0}} \left(\int_{-z_B}^{0} \sqrt{1 - z^2/z_B^2} + \int_0^{z_E} \sqrt{1 - z^2/z_E^2} \right)$$

$$u = 1 - z^2/z_B^2 \longleftrightarrow z = z_B (1-u)^{1/2}$$

$$du = -2z/z_B^2 \, dz = \frac{-2}{z_B} \sqrt{1-u} \, dz$$

$$J = -\sqrt{\tfrac{m}{2}} \sqrt{W_{\parallel 0}} \int_0^1 -z_B \, u^{1/2} (1-u)^{-1/2} du - \sqrt{\tfrac{m}{2}} \sqrt{W_{\parallel 0}} \, z_E \int_1^0 u^{1/2} (1-u)^{-1/2} du$$

$$= \sqrt{\tfrac{m}{2}} \int_0^1 u^{1/2} (1-u)^{-1/2} du \, \sqrt{W_{\parallel 0}} \, (z_B + z_E)$$

e. $\phi_0 \to \gamma \phi_0$, $B_0 \to \alpha B_0$ adiabatically.

$$\mu = \frac{W_{\perp 0}}{B_0} = \frac{W_{\perp 0}'}{\alpha B_0} \rightarrow W_{\perp 0}' = \alpha W_{\perp 0}$$

Assume z_B here means $|z_B|$.

$$J = \sqrt{W_{\parallel 0}} (z_E + z_B) = \sqrt{W_{\parallel 0}'} (z_E' + z_B')$$

$$\sqrt{W_{\parallel 0}} \left[\left(\frac{q}{W_{\parallel 0}} \frac{\phi_0}{L^2} \right)^{-1/2} + \left(\frac{W_{\perp 0}}{W_{\parallel 0}} \frac{R-1}{L^2} \right)^{-1/2} \right] = \sqrt{W_{\parallel 0}'} \left[\left(\frac{q \gamma \phi_0}{W_{\parallel 0}' L^2} \right)^{-1/2} + \left(\frac{\alpha W_{\perp 0}}{W_{\parallel 0}'} \frac{R-1}{L^2} \right)^{-1/2} \right]$$

$$W_{\parallel 0} \left(\frac{1}{\sqrt{q\phi_0}} + \frac{1}{\sqrt{W_{\perp 0}(R-1)}} \right) = W_{\parallel 0}' \left(\frac{1}{\sqrt{\gamma}\sqrt{q\phi_0}} + \frac{1}{\sqrt{\alpha}\sqrt{W_{\perp 0}(R-1)}} \right)$$

$$\sqrt{\gamma} \, W_{\parallel 0} \left(\frac{\sqrt{q\phi_0} + \sqrt{W_{\perp 0}(R-1)}}{\sqrt{\gamma}\sqrt{q\phi_0} + \sqrt{\alpha}\sqrt{W_{\perp 0}(R-1)}} \right) = W_{\parallel 0}'$$

For $z > 0$, ~~tra~~ new ~~trapping~~ unbound condition is $W_{\parallel 0}' > q\gamma\phi_0$

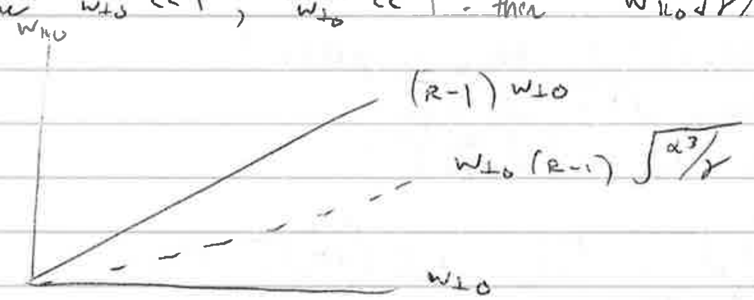
$$W_{\parallel 0} \sqrt{\gamma} \left(\frac{\sqrt{q\phi_0} + \sqrt{W_{\perp 0}(R-1)}}{\sqrt{\gamma}\sqrt{q\phi_0} + \sqrt{\alpha}\sqrt{W_{\perp 0}(R-1)}} \right) > q\gamma\phi_0$$

initially trapped: $W_{\parallel 0} < q\phi_0$

f. For $z < 0$, initially trapped: $\frac{W_{\parallel 0}}{W_{\perp 0}} < R-1$

Finally free: $\frac{W_{\parallel 0}'}{W_{\perp 0}'} > R-1 \rightarrow \frac{W_{\parallel 0}'}{\alpha W_{\perp 0}} > R-1 \rightarrow \sqrt{\frac{\gamma}{\alpha^3}} \frac{W_{\parallel 0}}{W_{\perp 0}} > R-1$

Assume $\frac{W_{\parallel 0}}{W_{\perp 0}} \ll 1$, $\frac{q\phi_0}{W_{\perp 0}} \ll 1$. then $W_{\parallel 0}\sqrt{\gamma/\alpha} = W_{\parallel 0}'$



if $\alpha^3/\gamma \lessapprox 1$, the wedge between the dashed and solid lines represents initially trapped and finally free (exiting $z < 0$ side)