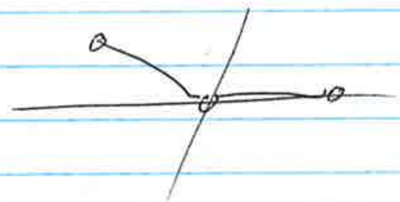
2014 Day 1 Q6

$\epsilon y'' + x^3 y' - 2xy = 0 \qquad x^3 \geq 0$, layer on left.

a. 

$\epsilon x^{-2} - 2x \sim 0 \rightarrow \epsilon \sim x^3$

$\delta = \epsilon^{1/3}$

b. $x^3 y' - 2xy = 0$

$y' - \frac{2}{x^2} y = 0 \qquad y = A e^{\int \frac{-2}{x^2} dx} = A e^{\frac{2}{x}}$

$y(1) = 1 \rightarrow y = e^{-2} e^{2/x}$

c. $x = \epsilon^{1/3} X \quad , \quad \frac{dy}{dx} = \frac{dY}{dX}\frac{dX}{dx} = \frac{1}{\epsilon^{1/3}} \frac{dy}{dx}$

$\epsilon y'' - 2xy = 0$

$\epsilon^{1/3} Y'' - 2\epsilon^{1/3} X Y = 0 \qquad Y'' - 2XY = 0$ Airy-like eqn.

$X \to 0$ ~~regular~~ ord, ~~Frobenius~~ Power : $y = \sum a_n x^{n+r}$

$\sum_{n=0} a_n (n+r)(n+r-1) x^{n+r-2} + \sum_{n=0} -2 a_n x^{n+r+1}$

$n-2 = j+1 \rightarrow j = n-3, \quad n = j+3$

$\sum_{j=-3} a_{j+3} (j+3+r)(j+2+r) x^{j+r+1}$ ~~oops, is ordinary point~~, use $r=0$

$j=-3: \quad a_0 r(r-1) = 0$

$j=-2: \quad a_1 r(r+1) = 0$

$j=-1: \quad a_2 (r+1)(r+2) = 0$

$\sum_{n=0} \left(a_{n+3}(n+r+3)(n+r+2) - 2a_n \right) x^{n+r+1} = 0$

$\sum_{n=0} a_n n(n-1) x^{n-2} + \sum_0 -2a_n x^{n+1}$

$\sum_{j=-3} a_{j+3} (j+3)(j+2) x^{j+1} + \sum_0 -2a_n x^{n+1}$

$j=-1: \quad a_2 \cdot 2 \cdot 1 = 0 \rightarrow a_2 = 0$

$\sum_{n=0} \{ a_{n+3}(n+3)(n+2) - 2a_n \} X^{n+1} = 0$

$a_{n+3} = \frac{2}{(n+2)(n+3)} a_n \qquad n \geq 0, \qquad a_0, a_1$ free, $\qquad a_2 = 0$

$a_0 = 1$ for B.C., $\rightarrow \qquad a_{3m} = \left(\frac{2}{9}\right)^m \frac{\Gamma(-1/3)}{\Gamma(m+1)\Gamma(m+2/3)} a_0$

$y(x) = \sum_{m=0} a_{3m} x^{3m}$

d. $Y'' - 2XY = 0$

inner

$y'' - 2xy = 0 \qquad y(x) = \int_c e^{xt} f(x)\, dt$

$\int_c (t^2 - 2x) f e^{xt} dt = 0$

$u = f \quad dv = x e^{xt}$

$du = f' \quad v = e^{xt}$

$-2 f e^{xt} \Big|_c + \int_c (t^2 f + 2f') e^{xt} dt = 0$

$f' + \frac{1}{2} t^2 f = 0 \rightarrow f = A e^{-\frac{1}{2}\int t^2 dt} = A e^{-\frac{1}{6}t^3}$

$e^{-\frac{1}{6}t^3} e^{xt} \Big|_c = 0$ when $t^3 \to +\infty \quad t_\infty = \lim_{R\to\infty} R e^{\frac{2\pi k}{3}}$

$y(x) = A \int_c e^{xt} e^{-\frac{1}{6}t^3} dt$

c. (misread instructions redo) $\quad X \to \infty \quad$ irrgpart -

$S'' + S'^2 - 2x \sim 0 \qquad (x \to \infty)$

$S'^2 \sim 2x, \quad S' \sim \pm\sqrt{2x}$

$Y \sim A_+ e^{\frac{2\sqrt{2}}{3} x^{3/2}} + A_- e^{-\frac{2\sqrt{2}}{3} x^{3/2}}$

$Y(X=0) = A_+ + A_- = 1$

matching: $\lim_{X\to\infty} Y = A_+ e^{\alpha^{3/2}} + A_- e \sim \lim_{x\to 0} Y_{in} = e^{-2} e^{2x^{-1}}$

$\rightarrow A_+ = e^{-2}, \quad A_- = 1 - e^{-2}$

e. $y(0) = 1 = \int_c A e^{-\frac{1}{6}t^3} dt \qquad$ should all work. —

f. ...