2015 ~~Waves~~ Day 2 Q1 Waves

a. Assume background ions, electrons cold.
with hot minority ions.

(n=1 dominant) Cyclotron power $\mathbb{P}_{abs} = \frac{\omega_p^2}{8\sqrt{\pi}} \frac{e^{-\lambda}}{\lambda W} \frac{1}{k_{\parallel}} |E_+|^2 e^{-\zeta_n^2}$

$N^2 = R, L$ for $\approx$ parallel propagation to B, where

$R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega+\Omega_s)}$, $L = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega-\Omega_s)}$

Since $\Omega_s > 0$ for ions, must have $N^2 = L$ for resonance.
with ω fixed, this shows $N^2 = \frac{k_{\parallel}^2 c^2}{\omega^2} \approx 1 - \frac{\omega_{pm}^2}{\omega(\omega - \Omega_M(z))}$

$k_{\parallel} \sim \sqrt{\frac{\omega_{pm}^2}{c^2}}\sqrt{\frac{\omega}{\omega-\Omega}}$ where $\omega = \Omega_0$, $\Omega = \Omega_0\left(1 + \frac{z}{l_0}\right)$

so $\frac{1}{k_{\parallel}} \sim \frac{c}{\omega_{pm}} \sqrt{\frac{\Omega_0}{\Omega_0(1-(1+z/l_0))}}^{-1} \sim \frac{c}{\omega_{pm}} \sqrt{\frac{-z}{l_0}}$

$\zeta_{n=1} = \frac{\omega - \Omega}{k_{\parallel} W} \Rightarrow \zeta_n^2 = \frac{(\Omega_0)^2}{\Omega_0^2}\left(\frac{-z}{l_0}\right)^2 \frac{c\,\Omega_0}{\omega_{pm} W}\left(\frac{-z}{l_0}\right)^{1/2}$

$\rightarrow$ ~~$\mathbb{P}_{abs} = \frac{\omega_p^2}{8\sqrt{\pi}} \frac{e^{-\lambda}}{\lambda W} \frac{c}{\omega_{pm}} \sqrt{\frac{-z}{l_0}}\, e^{-\frac{c}{W}\frac{\Omega_0}{\omega_{pm}}\left(\frac{-z_0}{l_0}\right)^{5/2}}$~~

$\omega - \Omega = \Omega_0[1 - (1 - z/L)] = \Omega_0(-z/L)$

$\zeta_1^2 = \frac{(\omega-\Omega)^2}{k_{\parallel}^2 W^2} = \frac{\Omega_0^2(-z/L)^2}{W^2}\left(\frac{c}{\omega_{pm}}\right)^2\left(\frac{-z}{L}\right)$

$\mathbb{P}_{abs} = \frac{\omega_p^2}{8\sqrt{\pi}} \frac{e^{-\lambda}}{\lambda W} \frac{c\,|E_+|^2}{\omega_{pm}} \sqrt{\frac{-z}{l_0}}\, e^{-\left(\frac{\Omega_0}{\omega_{pm}}\right)^2\left(\frac{c^2}{W^2}\right)\left(\frac{-z}{l_0}\right)^3}$

Assume $l_0 < 0$ wlog (if $l_0 > 0$, integrate instead $z: 0 \to -\infty$)

$P^{tot} = \int_0^{D} \mathbb{P}_{abs}(z)\,dz$ where D is the length of the plasma.
Though since this is a super exponential decay, the dominant contribution will only be from some small region at small z (though not necessarily peaked at z=0 in this case). So $\int_0^D \mathbb{P}_{abs}\,dz \approx \int_0^\infty \mathbb{P}_{abs}\,dz$.
~~Ignore prefactors~~, take $L = |l_0|$, then

$\mathbb{P}^{tot}_{abs} = (\quad) \int_0^\infty \sqrt{\frac{z}{L}}\, e^{-\left(\frac{\Omega_0}{\omega_{pm}}\right)^2\left(\frac{c}{W}\right)^2\left(\frac{z}{L}\right)^3} dz$

let $u = \frac{z}{L}\left(\frac{\Omega_0}{\omega_{pm}}\frac{c}{W}\right)^{2/3}$, $du = \frac{1}{L}dz\,(\quad)^{2/3}$

$\mathbb{P}^{tot} = (\quad) L M^{-1/3} M^{-2/3} \int_0^\infty u^{1/2} e^{-u^3} du$

a. $$\mathbb{P}^{tot} = \frac{\omega_p^2}{8\sqrt{\pi}} \frac{e^{-\lambda}}{\lambda} \frac{c}{\cancel{W}} \frac{|E_+|^2}{\omega_{pm}} L \left(\frac{\cancel{\omega_{pm}}\ \cancel{W}}{\Omega_0 c}\right) \int_0^\infty u^{1/2} e^{-u^3}\, du$$

let $s = u^3$, $ds = 3u^2\, du$,

$$\int_0^\infty u^{1/2} e^{-s}\left(\frac{ds}{3u^2}\right) = \frac{1}{3}\int u^{-3/2} e^{-s}\, ds = \frac{1}{3}\int s^{-1/2} e^{-s}\, ds$$

$$= \frac{1}{3}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\rightarrow \mathbb{P}^{tot} = \frac{\omega_p^2}{16}\frac{e^{-\lambda}}{\lambda}\frac{|E_+|^2 L}{\Omega_0}$$

b. Consider the integrand with $u = \frac{z}{|L_0|}$, $\alpha^2 = \left(\frac{\Omega_0}{\omega_{pm}}\frac{c}{W}\right)^2$

$$\sim f(u) = u^{1/2} e^{-\alpha^2 u^3}$$

$$f'(u) = \frac{1}{2} u^{-1/2} \cancel{e^{-\alpha^2 u^3}} - 3\alpha^2 u^{5/2} \cancel{e^{-\alpha^2 u^3}} = 0$$

$$\frac{1}{2} - 3\alpha^2 u^3 = 0 \rightarrow u_0^3 = \frac{\alpha^2}{6}$$

Taylor expand around U_0 to get a characteristic length:

$$f''(u) = \left(-\frac{1}{4} u^{-3/2} - \frac{15\alpha^2}{2} u^{3/2}\right) e^{-\alpha^2 u^3} + \left(\frac{1}{2} u^{-1/2} - 3\alpha^2 u^{5/2}\right)\left(-3\alpha^2 u^2\right) e^{-\alpha^2 u^3}$$

at $u = u_0$:

$$\left[-\frac{1}{4}\left(\frac{\alpha^2}{6}\right)^{-1/2} - \frac{15\alpha^2}{2}\left(\frac{\alpha^2}{6}\right)^{1/2}\right] e^{-\alpha^4/6} + \left[\frac{1}{2}\left(\frac{\alpha^2}{6}\right)^{-1/6} - 3\alpha^2\left(\frac{\alpha^2}{6}\right)^{5/6}\right]\left(-3\alpha^2\left(\frac{\alpha^2}{6}\right)^{2/3}\right) e^{-\alpha^4/6}$$

This looks like a mess, but just suppose this is $f''(U_0)$, then

$$f(u_0+\Delta u) \approx f(u_0) + \Delta u\, \cancel{f'(u_0)}^{0} + \frac{1}{2}\Delta u^2 f''(u_0)$$

Define absorption zone so that $\frac{f(u_0+\Delta u)}{f(u_0)} = \frac{\frac{1}{2}\Delta u^2 f''(u_0)}{f(u_0)} \sim \epsilon \ll 1$

then we can solve for U_0 with this condition.

A simpler estimate would be to forget about the $\sqrt{u}$ coefficient and simply look at the exponential decay which goes as $\cancel{\frac{\alpha^2}{L_0^3}}$ $\frac{L_0^3}{\alpha^2} \sim a_{abs}$

but α dimless...

c. $L_0 \sim$ minor radius $\sim 1\,m$

$$\alpha^2 = \left(\frac{\Omega_0}{\omega_{pm}}\frac{c}{W}\right)^2 = \frac{(2B/mc)^2}{4\pi n e^2/m}\left(\frac{c}{\sqrt{T/m}}\right)^2 = \frac{B^2/8\pi}{\frac{1}{2} n \cancel{mc^2}}\frac{\cancel{c^2}}{T/\cancel{m}} = \frac{B^2}{4\pi n T}$$

use $B^n \sim 5\,T$, $n \sim 10^{20}\, m^{-3}$, $T \sim 10$ keV

~~$\alpha^2 \sim$~~ convert back to SI...

$$\alpha^2_{SI} = \left(\frac{2B}{m}\right)^2\left(\frac{m\epsilon_0}{ne^2}\right)\frac{e^2}{T/m} = \frac{B^2\epsilon_0 c^2}{nT} =$$

$$= \left(\frac{B^2/2\mu_0}{nT}\right)\frac{2}{\mu_0} = \frac{2}{\mu_0}\beta^{-1}$$ units must be wrong...