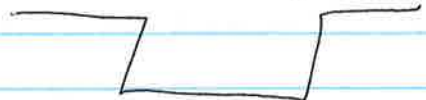
2015 Day 1 Q2 Waves

a. $\epsilon = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{k^2}\int_L \frac{f_0'(v)}{v - \frac{\omega}{k}}\, dv$

$f_0 = A +$  $\Big] \frac{n_H}{2\bar{v}}\frac{1}{n_0}$

$f_0' = \frac{1}{2\bar{v}}\frac{n_H}{n_0}\left[-\delta(v+\bar{v}) + \delta(v-\bar{v})\right]$

Suppose $\frac{\omega}{k} \neq \bar{v}$, then the Landau part does not contribute.
(equivalently, treat the edge case later.

$\epsilon = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{k^2}\frac{n_H}{2\bar{v}n_0}\left[\frac{1}{\bar{v}-\frac{\omega}{k}} - \frac{1}{-\bar{v}-\frac{\omega}{k}}\right]$

$\left(\frac{1}{\bar{v}-\frac{\omega}{k}} + \frac{1}{\bar{v}+\omega/k} = \frac{2\bar{v}}{\bar{v}^2 - \left(\frac{\omega}{k}\right)^2}\right)$

$\epsilon = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{k^2}\frac{n_H}{n_0}\left(\frac{1}{\bar{v}^2 - \left(\frac{\omega}{k}\right)^2}\right)$

b. $\epsilon = 1 - \frac{\omega_{pi}^2}{\omega^2} - \omega_{pe}^2\frac{n_H}{n_0}\frac{1}{(k^2\bar{v}^2 - \omega^2)} = 0$

Rewrite $\omega^2(k^2\bar{v}^2 - \omega^2) - \omega_{pi}^2(k^2\bar{v}^2 - \omega^2) - \frac{\omega_{pe}^2 n_H \omega^2}{n_0} = 0$

$\omega^4 - \omega^2\left(k^2\bar{v}^2 + \omega_{pi}^2 - \frac{n_H}{n_0}\omega_{pe}^2\right) + \omega_{pi}^2 k^2 \bar{v}^2 = 0$

Define $\Delta = \omega_{pi}^2 - \frac{n_H}{n_0}\omega_{pe}^2$

$\rightarrow 2\omega^2 = k^2\bar{v}^2 + \Delta \pm \sqrt{(k^2\bar{v}^2 + \Delta)^2 - 4\omega_{pi}^2 k^2\bar{v}^2}$

$= (k^2\bar{v}^2 + \Delta)\left[1 \pm \sqrt{1 - \frac{4\omega_{pi}^2 k^2\bar{v}^2}{(k^2\bar{v}^2 + \Delta)^2}}\right]$

If $\frac{4\omega_{pi}^2 k^2\bar{v}^2}{(k^2\bar{v}^2 + \Delta)^2} > 1$, there will be an unstable mode.

Note $\mathrm{sgn}(\Delta)$ depends on size of $\frac{n_H}{n_0}$

c. $W = \omega I = \omega L_\omega = \frac{|E|^2}{16\pi}\left[e^*\cdot\partial_\omega(\omega\epsilon)\cdot e + \cancelto{0}{\frac{c^2}{\omega^2}|k\times e|^2}\right]$

$\partial_\omega(\omega\epsilon) = 1 + \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{k^2}\frac{n_H}{n_0}\left(\frac{1}{\bar{v}^2 - \left(\frac{\omega}{k}\right)^2}\right) - \frac{2\omega^2}{k^2}\left(\quad\right)$

$\cancel{\rightarrow W = \frac{|E|^2}{16\pi}\left[1 + \frac{\omega_{pi}^2}{\omega^2} - \left(1 + \frac{2\omega^2}{k^2}\right)\right.}$

$W = \frac{|E|^2}{16\pi}\left[1 + \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{k^2}\frac{n_H}{n_0}\left(\frac{1}{\bar{v}^2 - \frac{\omega^2}{k^2}}\right) - \frac{2\omega^2}{k^2}\frac{\omega_{pe}^2 n_H}{k^2 n_0}\left(\quad\right)^2\right]$

e, $$W = \frac{|\mathcal{E}|^2}{16\pi}\left[1 + \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2 n_H}{k^2 n_0}\left[\frac{1}{\bar{v}^2 - \left(\frac{\omega}{k}\right)^2}\right]\right]\left[1 + \frac{2\omega^2}{k^2}\frac{1}{\bar{v}^2-\left(\frac{\omega}{k}\right)^2}\right]$$

W can be negative, especially for $\left(\frac{\omega}{k}\right)$ near $\bar{v}$ ...

Now for the $|v| = \bar{v}$ edgecase. For $\operatorname{Im}\omega \leq 0$, need to include Landau contribution in dispersion:

(assume $k>0$) $$\epsilon = \epsilon_{previous} - i\pi\frac{\omega_{pe}^2}{k^2} f_0'\left(\frac{\omega}{k}\right)\begin{cases}1 & \operatorname{Im}\omega = 0\\ 2 & \operatorname{Im}\omega < 0\end{cases}$$

~~At $\frac{\omega}{k} = \pm\bar{v}$,~~ Probably $\bar{v}$ real.

At $\frac{\omega}{k} = \pm\bar{v}$, get $-i\pi\frac{\omega_{pe}^2}{k^2}\left(\frac{1}{2\bar{v}}\frac{n_H}{n_0}\right)\left(\delta(v-\bar{v}) - \delta(v+\bar{v})\right)$

so it looks like this spikes to ∞. But the PV part of the dispersion also spikes to infinity here. This probably corresponds to the unphysical distribution (discontinuous)..