2015 Day 1 Q6 Asymptotics

a. $\frac{1}{\Lambda} y^{(4)} + x y'' + \gamma y = 0$

$\frac{1}{\Lambda} S'^4 + x S'^2 + \gamma \sim 0$

1. $S'^2 \left(\frac{S'^2}{\Lambda} + x \right) \sim 0 \to S' \sim 0$ weak

$S' \sim \pm i \sqrt{x \Lambda}$ ✓

2. $S' \sim e^{i\pi k/4} (-1)^{1/4} (\gamma \Lambda)^{1/4}$ weak

3. $S'^2 \sim \pm i \sqrt{\gamma / x}$ ✓

verify they obey $\frac{S''}{S'^2} \ll 1$, $x \to \infty$: $\frac{x^{-1/2}}{x} \to 0$ ✓ and $\frac{x^{-3/2}}{x} \to 0$ ✓.

b. $y(x) = \int_c f(t) e^{xt} \, dt$

$\int_c \left(\frac{1}{\Lambda} t^4 + x t^2 + \gamma \right) e^{xt} f(t) \, dt = 0$

integrate by parts: $u = t^2 f$, $dv = x e^{xt}$

$du = 2tf + t^2 f'$, $v = e^{xt}$

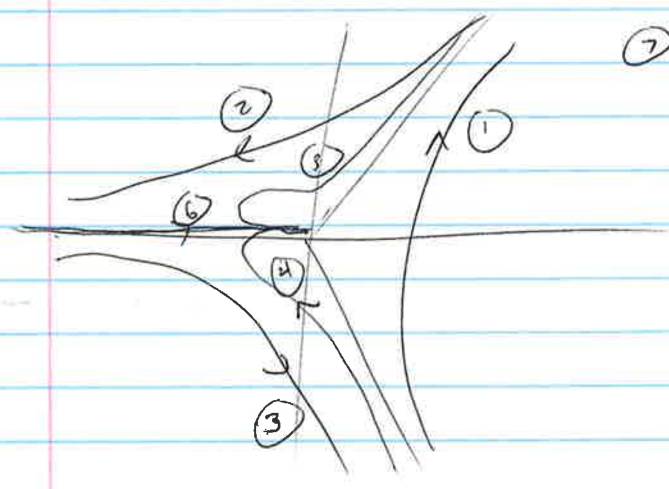
$\int_c dt \left(\frac{1}{\Lambda} t^4 - 2t + \gamma \right) f(t) - t^2 f'(t) \; e^{xt} + t^2 f(t) e^{xt} \Big|_c = 0$

$\to \log f = \frac{1}{3\Lambda} t^3 - 2 \log t - \gamma / t \to f(t) = \frac{A e^{\frac{1}{3\Lambda} t^3 - \gamma/t}}{t^2}$

$\longrightarrow e^{xt} e^{\frac{1}{3\Lambda} t^3 - \frac{\gamma}{t}} \Big|_c = 0$

c. vanishes for $|t| \to \infty$, $\arg(t^3) = \pi \to \arg t = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

for $\gamma < 0$, approaching $t = 0$ from $t < 0$ will work also.



⑦ around the origin? Should there be a cut?

$$t^2 = \frac{\alpha}{2\Lambda}\left(\begin{matrix} -\frac{1}{2}\gamma^2/x^2 \\ -2 + \frac{1}{2}\frac{\gamma^2}{x^2}\end{matrix}\right) \to t \sim \pm i\frac{\gamma}{x}\sqrt{\frac{x}{2}\Lambda}$$

$$t \sim i\sqrt{\frac{x}{\Lambda}}$$

$(\gamma < 0)$

$t = \sqrt{x^2+y^2}$

$\partial_t x = \frac{x}{\sqrt{x^2+y^2}}$

d. $\phi(x,t) = xt + \frac{1}{3\Lambda}t^3 - \frac{\gamma}{t}$

$\phi'(x,t) = x + \frac{t^2}{\Lambda} + \frac{\gamma}{t^2} = 0$

occurs at $\frac{t^4}{\Lambda} + xt^2 + \gamma = 0$

$2\Lambda t^2 = -x \pm \sqrt{x^2-\gamma^2} = x\left(-1 \pm \left(1 - \frac{1}{2}\frac{\gamma^2}{x^2}\right)\right)$

$\cancel{t = \pm\sqrt{\frac{-x}{2\Lambda} \pm \frac{\sqrt{x^2-\gamma^2}}{2\Lambda}}}$ or

$\frac{t}{x^2+y} \to \frac{-2x}{2\Lambda(x^2+y^2)^2}$

$\frac{1}{\sqrt{x^2+y^2}} \to \frac{-2x\frac{1}{2}}{(x^2+y^2)^{3/2}}$

$t^4 + \Lambda x t^2 + \Lambda\gamma = 0$

$t = \pm\sqrt{\frac{-\Lambda x \pm \sqrt{(\Lambda x)^2 - (\Lambda\gamma)^2}}{2}}$

suppose $|x|$ large: $\Lambda x^2 \gg 1$ then

$\frac{1}{\Lambda}t^4 + xt^2 + \gamma = 0$

$\frac{2}{\Lambda}t^2 = -x \pm \sqrt{x^2-\gamma^2} = x\left(-1 \pm \sqrt{1-\frac{\gamma^2}{x^2}}\right)$

$= x\left(-1 \pm \left(1 - \frac{1}{2}\frac{\gamma^2}{x^2}\right)\right)$

$= x\left\{\begin{matrix} -1/2\ \gamma^2/x^2 \\ -2 + \frac{1}{2}\gamma^2/x^2\end{matrix}\right.$

$t^2 = -\frac{1}{4}\gamma^2 x\Lambda,\ -\Lambda x + \frac{1}{4}\frac{\Lambda\gamma^2}{x}$

$t \simeq \pm\frac{i\gamma}{2}\sqrt{x\Lambda},\ \pm i\sqrt{x\Lambda}$

$\nabla\phi = x\frac{\vec{t}}{t^2} + \frac{2t}{\Lambda} - \frac{\vec{t}\gamma}{t^{3/2}}$