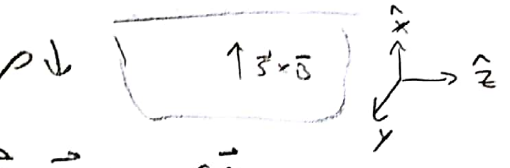
2007 II : Q2 MHD

(a.) $\nabla \times B = \mu_0 \vec{J} \Rightarrow \boxed{\vec{J} = \frac{1}{\mu_0} \nabla \times \vec{B}}$



The force balance is given by $\vec{J} \times \vec{B} = -\rho \vec{g}$

$$\Rightarrow \vec{J} \times \vec{B} = \rho g \hat{x}$$

so $\frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{B} = \rho g \hat{x}$

$B(x) \frac{\partial}{\partial z} B(x) = 0$

using $\nabla(\vec{B} \cdot \vec{B}) = 2\vec{B} \times (\nabla \times \vec{B}) + 2\cancel{(\vec{B} \cdot \nabla)\vec{B}}^{\,0}$ we get

$$\boxed{0 = \frac{\partial}{\partial x}\left(\frac{B^2(x)}{2\mu_0}\right) + \rho g}$$ magnetic pressure balances gravity

(b.) $\delta W = \frac{1}{2}\int_V d^3x \left[\gamma \cancel{P}^{\,P=0} |\nabla \cdot \vec{\xi}|^2 - \vec{\xi}^* \cdot \cancel{\nabla(\vec{\xi} \cdot \nabla P)}^{\,P=0} + \frac{1}{\mu_0}|Q|^2 \right.$

$$\left. - \vec{\xi}^* \cdot \vec{J} \times \vec{Q} + \vec{\xi}^* \cdot \vec{g}\, \nabla \cdot (\rho \vec{\xi}) \right]$$

Assume $\vec{\xi} = \vec{\xi}(x) e^{iky}$. $|Q|^2 > 0$ is the energy required to bend field lines, so for worst case we also have $|Q|^2 = 0$

Then $\delta W = \frac{1}{2}\int_V d^3x \left[-\vec{\xi}^* \cdot \vec{J} \times \vec{Q} + \vec{\xi}^* \cdot \vec{g}\, \nabla \cdot (\rho \vec{\xi}) \right]$

$$= \frac{1}{2}\int_V d^3x \left[-\vec{\xi}^*(x) e^{-iky} \cdot \vec{J} \times \left[\nabla \times \left(\vec{\xi}(x) e^{iky} \times B(x)\hat{z}\right)\right] \right.$$

$$\left. + \vec{\xi}^*(x) e^{-iky} \cdot \vec{g}\, \nabla \cdot \left(\rho \vec{\xi}(x) e^{iky}\right) \right]$$

gravity term: $\vec{\xi}^* \cdot \vec{g}\left[\rho(\nabla \cdot \vec{\xi}) + \vec{\xi} \cdot \nabla \rho\right]$

current term: $-\vec{\xi}^* \cdot \frac{1}{\mu_0}(\nabla \times \vec{B}) \times (\nabla \times (\vec{\xi} \times \vec{B}))$ 0 since $\frac{\partial}{\partial z}\vec{\xi} = 0$

$$= -\vec{\xi}^* \cdot \frac{1}{\mu_0}(\nabla \times \vec{B}) \times \left[-\vec{B}(\nabla \cdot \vec{\xi}) + \cancel{(\vec{B} \cdot \nabla)\vec{\xi}} - (\vec{\xi} \cdot \nabla)\vec{B}\right]$$

It would really be nice to assume incompressibility here...

$$\delta W = \frac{1}{2}\int dx\int dy\int dz \; -\vec{\xi}^{*}(x)e^{-iky}\cdot\frac{1}{\mu_0}\overset{(1)}{(\nabla\times\vec{B})}\times\Big[\overset{(2)}{-\vec{B}(\nabla\cdot\vec{\xi}(x)e^{iky})}$$

$$\overset{(3)}{-\xi_x(x)e^{iky}\frac{\partial}{\partial x}B(x)\hat{z}}\Big] + \vec{\xi}^{*}e^{-iky}\cdot\vec{g}\big[\underset{(4)}{\rho(\nabla\cdot\vec{\xi}(x)e^{iky})} + \underset{(5)}{\vec{\xi}\cdot\nabla\rho}\big]$$

(1)
$$\nabla\times\vec{B} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 0 & 0 & B(x)\end{vmatrix} = \frac{\partial}{\partial x}B(x)\hat{y}$$

(2)
$$-\vec{B}(\nabla\cdot\vec{\xi}(x)e^{iky}) = -B(x)\left[\frac{\partial}{\partial x}\xi(x) + \xi(x)ik\right]e^{iky}\hat{z}$$

$$\frac{\partial B^2}{\partial x} = 2\frac{\partial B}{\partial x}B$$

$$(1)\times(2) = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ & * & \\ & & *\end{vmatrix} = -\frac{\partial B}{\partial x}B\left(\frac{\partial\xi}{\partial x} + ik\xi\right)e^{iky}\hat{x}$$

$$(1)\times(3) = -\frac{\partial B}{\partial x}\left(\xi_x e^{iky}\frac{\partial B}{\partial x}\right)\hat{x} = -\xi_x e^{iky}\left(\frac{\partial B}{\partial x}\right)^2\hat{x}$$

(4) $-\rho g\left(\frac{\partial\xi}{\partial x} + ik\xi\right)e^{iky}\hat{x}$ (5) $-g\left[\xi_x e^{iky}\frac{\partial\rho}{\partial x}\right]\hat{x}$

(1)×(2) and (4) cancel exactly since $-\rho g - \frac{1}{\mu_0}\frac{\partial}{\partial x}\left(\frac{B^2}{2}\right) = 0$

so $\delta W = \frac{1}{2}\int_V d^3x \left[-|\xi_x|^2\frac{1}{\mu_0}\left(\frac{\partial B}{\partial x}\right)^2 - |\xi_x|^2 g\frac{\partial\rho}{\partial x}\right]$

need $\delta W \geq 0$ for stability so

$$\cancel{|\xi_x|^2}\frac{1}{\mu_0}\left(\frac{\partial B}{\partial x}\right)^2 + \cancel{|\xi_x|^2}g\frac{\partial\rho}{\partial x} \geq 0$$

$$\boxed{g\frac{\partial\rho}{\partial x} \leq -\frac{1}{\mu_0}\left(\frac{\partial B}{\partial x}\right)^2}$$ for stability

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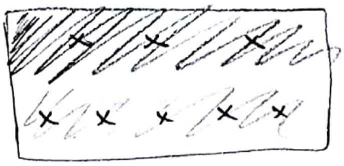
Does this make sense?

$\downarrow\rho g$ 

Density gradiant decrease less than increase in magnetic pressure gradient.