2008 II : Q5 MHD

(a.) $H = \int \vec{A}\cdot\vec{B}\, dV$, $\vec{B} = \nabla\times\vec{A}$

H is gauge invariant so $\frac{d\vec{A}}{dt} = -\vec{E} - \nabla\phi$

$$\frac{dH}{dt} = \int \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}\, dV$$

$$= \int dV \; -\vec{E}\cdot\vec{B} - \nabla\phi\cdot\vec{B} - A\cdot\nabla\times\vec{E}$$

$$-A\cdot\nabla\times\vec{E} = \nabla\cdot(\vec{A}\times\vec{E}) - \vec{E}\cdot\nabla\times\vec{A}$$
$$= \nabla\cdot(\vec{A}\times\vec{E}) - \vec{E}\cdot\vec{B}$$

$$-\nabla\phi\cdot\vec{B} = -\nabla\cdot(\phi\vec{B}) + \phi\cancelto{0}{\nabla\cdot\vec{B}}$$

$$\frac{dH}{dt} = \int dv \; -2\vec{E}\cdot\vec{B} + \nabla\cdot(\vec{A}\times\vec{E} - \phi\vec{B})$$

$$= \int dV \; \underbrace{-2\vec{E}\cdot\vec{B}}_{=0 \text{ from Ohm's Law. } \vec{E} = -\vec{U}\times\vec{B}} + \int ds \; \underbrace{\vec{A}\times\vec{E} - \phi\vec{B}}_{=0,\text{ vanishing boundary terms}}$$

$\therefore \boxed{\frac{dH}{dt} = 0} \Rightarrow$ helicity is conserved

(b.) $$\delta W = \frac{1}{2}\int dV \cdot \Big[\underbrace{\frac{Q_\perp^2}{2}}_{1} + \underbrace{\frac{B^2}{\mu_0}(\nabla\cdot\xi_\perp + 2\vec{\xi}_\perp\cdot\vec{\kappa})^2}_{2} + \underbrace{\gamma p(\nabla\cdot\vec{\xi})^2}_{3}$$
$$\underbrace{-2(\vec{\xi}_\perp\cdot\nabla p)(\vec{\xi}_\perp^{*}\cdot\vec{\kappa})}_{4} \underbrace{- \mu(\vec{r})(\xi_\perp^{*}\times\vec{B})\cdot Q_\perp}_{5} \Big]$$

1, 2 and 3 are always positive. thus always stabilizing

1: bending B   2: compressing B   3: compressing plasma

4 and 5 can have either sign → potentially destabilizing

4: pressure-driven instability.   5: current-driven instability.

(c.) Ideal force-free equilibria implies $\vec{E} = -\vec{u}\times\vec{B}$, $\vec{J}\times\vec{B} = 0$

So we can drop term ④ because $\nabla P = 0$

the only unstabilizing term is $-N(\vec{r})(\vec{\xi}_\perp^* \times \vec{B})\cdot\vec{Q}_\perp$, so a sufficient condition for stability is

$$\delta W_S = \frac{1}{2}\int dV \; -N(\vec{r})(\vec{\xi}_\perp^* \times \vec{B})\cdot\vec{Q}_\perp \geq 0$$

$$\vec{Q} = \nabla\times(\vec{\xi}_\perp \times \vec{B}) = \nabla\times\vec{A}_{\perp 1} \quad \text{so } \vec{A}_{\perp 1} = \vec{\xi}_\perp \times \vec{B} \quad \text{so } \vec{A}_{\parallel} = 0$$

$$\text{so } \delta W_S = \frac{1}{2}\int dV \; -N(\vec{r})\left[\vec{A}_{\perp 1}\cdot\vec{B}_{\perp 1}\right]$$

in part (a) it states "no interceptions of the magnetic field" so $H = \int dV \; \vec{A}\cdot\vec{B} = 0.$

If $N(\vec{r}) = \text{const.}$, $(N \propto J_{\parallel}/B)$ then we have

$$\delta W_S = -\frac{J_{\parallel}}{B}\int dV \; \vec{A}_{\perp 1}\cdot\vec{B}_{\perp 1} = -\frac{J_{\parallel}}{B}\int dV \; \vec{A}_1\cdot\vec{B}_1 = 0$$

(since the Helicity of the pertubations is also 0)

So a sufficient stability condition is $\underline{N(\vec{r}) = \text{const.}}$