2010 II: Q2 GPP

a.) $n(x,t) = \int_{-\infty}^{\infty} dv\, f(x,v,t) = \int_{v_-(x,t)}^{v_+(x,t)} A\, dv = A\left[v_+(x,t) - v_-(x,t)\right]$ ✓

$n(x,t)\, V(x,t) = \int_{v_-}^{v_+} dv\ Av = \frac{A}{2} v^2 \Big]_{v_-}^{v_+} = \frac{1}{2} A\left[v_+^2(x,t) - v_-^2(x,t)\right]$

$P(x,t) = m \int_{v_-}^{v_+} dv\, \left[v - V(x,t)\right]^2 A = mA \int_{v_-}^{v_+} dv\ v^2 - 2vV + V^2$ ✓

$= mA\left[\frac{1}{3}v^3 - v^2 V + V^2 v\right]_{v_-}^{v_+}$

$V = \frac{1}{2}\frac{v_+^2 - v_-^2}{v_+ - v_-} = \frac{1}{2}(v_+ + v_-)$

$= mA\left[\frac{1}{3}v_+^3 - v_-^3 - (v_+^2 - v_-^2)\frac{1}{2}\frac{v_+^2 - v_-^2}{(v_+ - v_-)} + \frac{1}{4}\left(\frac{v_+^2 - v_-^2}{v_+ - v_-}\right)^2 (v_+ - v_-)\right]$

$= \frac{mA}{12}\left[4(v_+^3 - v_-^3) - 6\frac{(v_+^2 - v_-^2)^2}{(v_+ - v_-)} + 3\frac{(v_+^2 - v_-^2)^2}{(v_+ - v_-)}\right]$

$= \frac{mA}{12}\left[4(v_+^3 - v_-^3) - 3(v_+^2 - v_-^2)(v_+ + v_-)\right]$ ✓

$= \frac{mA}{12}\left[v_+^3 - 3v_+^2 v_- + 3v_-^2 v_+ - v_-^3\right] = \frac{1}{2} mA\left[v_+(x,t) - v_-(x,t)\right]^3$

b.) $\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{e}{m}E\frac{\partial}{\partial v}\right] f(x,v,t) = 0$

$\frac{\partial f}{\partial v} = A\left[\delta(v - v_+) - \delta(v - v_-)\right]$

0th moment: $\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial t}f + v\frac{\partial}{\partial x}f - \frac{e}{m}E\frac{\partial}{\partial v}f\right)dv$

$\frac{\partial}{\partial t}\cancel{A}\left[v_+ - v_-\right] + \frac{\partial}{\partial x}\left[\frac{1}{2}\cancel{A}(v_+^2 - v_-^2)\right] = 0$ ✓

1st moment: $\frac{\partial}{\partial t}\left[\frac{A}{2}(v_+^2 - v_-^2)\right] + \frac{\partial}{\partial x}\int_{v_-}^{v_+} dv\ v^2 A = \frac{e}{m}E\int_{-\infty}^{\infty} dv\ v\frac{\partial f}{\partial v}$

$\frac{\partial}{\partial t}\left[\frac{1}{2}(v_+^2 - v_-^2)\right] + \frac{\partial}{\partial x}\left[\frac{1}{3}(v_+^3 - v_-^3)\right] = \frac{e}{m}E(v_+ - v_-)$ ✓

missing a (-) sign?

c.) $\frac{\partial v_+}{\partial t} - \frac{\partial v_-}{\partial t} + v_+ \frac{\partial v_+}{\partial x} - v_- \frac{\partial v_-}{\partial x} = 0 \Rightarrow \partial_t v_+ + v_+ \partial_x v_+ = \partial_t v_- + v_- \partial_x v_-$

$$v_+ \frac{\partial v_+}{\partial t} - v_- \frac{\partial v_-}{\partial t} + v_+^2 \frac{\partial v_+}{\partial x} - v_-^2 \frac{\partial v_-}{\partial x} = -\frac{e}{m} E (v_+ - v_-)$$

$$v_+ (\partial_t v_+ + v_+ \partial_x v_+) - v_- (\partial_t v_- + v_- \partial_x v_-) = -\frac{e}{m}(v_+ - v_-) E$$

$\underline{\partial_t v_+ + v_+ \partial_x v_+ = -\frac{e}{m} E}$ ✓ and $\underline{\partial_t v_- + v_- \partial_x v_- = -\frac{e}{m} E}$ ✓

d.) $Q(x,t) = m \int_{-\infty}^{\infty} dv \, [v - V(x,t)]^3 f(x,v,t)$

$$= m \int_{v_-}^{v_+} dv \, A [v^3 - 3v^2 V - 3V^2 v - V^3]$$

$$= mA \Big[\tfrac{1}{4}(v_+^4 - v_-^4) - (v_+^3 - v_-^3) \tfrac{1}{2}(v_+ + v_-)$$

$v_+^3 + 3v_+^2 v_- + 3v_-^2 v_+ + v_-^3$

$v_+^2 - 2v_+v_- + v_-^2$

$$- 3 \tfrac{1}{4}(v_+ + v_-)^2 \tfrac{1}{2}(v_+^2 - v_-^2) - \tfrac{1}{8}(v_+ + v_-)^3 (v_+ - v_-) \Big]$$

$$= 4mA \Big[\cancel{v_+^4} - \cancel{v_-^4} - 4v_+^4 - \underline{4v_+^3 v_-} + 4v_-^3 v_+ + 4v_-^4 - 3v_+^4 + \cancel{3v_+^2 v_-^2}$$

$$-6v_- v_+^3 + 6v_-^3 v_+ - \cancel{3v_-^2 v_+^2} + 3v_-^4 - \cancel{v_+^4} - \underline{3v_+^3 v_-}$$

$$- \cancel{3v_-^2 v_+^2} - v_-^3 v_+ + \cancel{v_-^4} + \cancel{3v_+^2 v_-^2} + 3v_+ v_-^3 + v_- v_+^3 \Big]$$

BLERG. Try to exploit $f' = A[\delta(v - v_+) - \delta(v - v_-)]$

IBP: $u = f$ $\quad v = \frac{1}{4}[v - V]^4$

$du = f' dv$ $\quad dv = [v - V]^3 dv$

$V = \frac{1}{2}(v_+ + v_-)$

$$\Rightarrow Q = -m \int_{-\infty}^{\infty} \frac{A}{4} [v - V]^4 (\delta(v - v_+) - \delta(v - v_-)) \, dv$$

$$= -\frac{mA}{4} \left[(v_+ - \tfrac{1}{2}v_+ - \tfrac{1}{2}v_-)^4 - (v_- - \tfrac{1}{2}v_+ - \tfrac{1}{2}v_-)^4 \right]$$

$$= -\frac{mA}{4} \left[(\tfrac{1}{2}v_+ - \tfrac{1}{2}v_-)^4 - (\tfrac{1}{2}v_- - \tfrac{1}{2}v_+)^4 \right] = 0 \quad ✓$$