2010 II : Q4 MHD

1.) In general, $W = \int dV \left(\frac{B^2}{2\mu_0} + \frac{\epsilon_0 E^2}{2} \right)$. We drop $\vec{E}$ because $\frac{E}{B} << 1$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \Rightarrow kE \sim \frac{\omega}{c} B$$

$$\Rightarrow \frac{E}{B} \sim \frac{\omega}{kc} \sim \frac{v_A}{c} << 1$$

So we need to keep $\frac{\epsilon_0 E^2}{2}$ for relativistic plasmas.

2.) $\delta \vec{B} \equiv \nabla \times (\vec{\xi} \times \vec{B})$ $\quad \delta W = \int dV \frac{2B\delta B}{2\mu_0} = \int dV \frac{1}{\mu_0} \vec{B} \cdot (\nabla \times (\vec{\xi} \times \vec{B}))$

$$\delta W = \frac{1}{\mu_0} \int dV \left[\nabla \cdot (\vec{B} \times (\vec{\xi} \times \vec{B})) + (\vec{\xi} \times \vec{B}) \cdot \underbrace{(\nabla \times \vec{B})}_{= \mu_0 \vec{J}} \right]$$

$$\delta W = \frac{1}{\mu_0} \left[\cancel{\int dS \, (\vec{B} \times (\vec{\xi} \times \vec{B}))}^{\,0} + \int dV \, (\vec{\xi} \times \vec{B}) \cdot \mu_0 \vec{J} \right]$$

$$\delta W = \int dV (\vec{\xi} \times \vec{B}) \cdot \vec{J} = \int dV \, \vec{\xi} \cdot (\vec{B} \times \vec{J})$$

$\therefore$ requiring $\delta W = 0$ for any $\vec{\xi} \Rightarrow \vec{B} \times \vec{J} = 0$.

3.) Variational principle: $\delta(W - \lambda K) = 0$

minimize W while keeping K cons

$$\delta(W - \lambda K) = \int dV \left[\frac{\vec{B} \cdot \delta \vec{B}}{\mu_0} - \lambda \delta A \cdot \vec{B} - \lambda \vec{A} \cdot \delta \vec{B} \right]$$

note: $\delta \vec{B} = \nabla \times \delta \vec{A}$

$$= \int dV \left[\frac{\vec{B}}{\mu_0} \cdot (\nabla \times \delta \vec{A}) - \lambda \vec{A} \cdot (\nabla \times \delta \vec{A}) - \lambda \delta \vec{A} \cdot \vec{B} \right]$$

$$= \int dV \left[\frac{1}{\mu_0} \cancel{\nabla \cdot (\delta \vec{A} \times \vec{B})}^{\,0} + \frac{1}{\mu_0} \delta \vec{A} \cdot (\nabla \times \vec{B}) - \lambda \cancel{\nabla \cdot (\delta \vec{A} \times \vec{A})}^{\,0} - \lambda \delta \vec{A} \cdot (\nabla \times \vec{A}) - \lambda \delta \vec{A} \cdot \vec{B} \right]$$

$$= \int dV \, \delta \vec{A} \cdot \vec{J} - 2\lambda \, \delta \vec{A} \cdot \vec{B} = \int dV \, \delta \vec{A} \cdot (\vec{J} - 2\lambda \vec{B})$$

This implies that $\vec{J} - 2\lambda \vec{B} = 0$ minimizes δW for any $\vec{\delta A}$.

Is this force-free?

$$\nabla P = \vec{J} \times \vec{B} = 2\lambda \vec{B} \times \vec{B} = 0 \implies \text{force-free!}$$

4.) (3) has constant helicity but (2) doesn't

(3) has $\lambda = \text{const}$ so that $\vec{J} \propto \vec{B}$.

(2) does not have $\vec{J} \propto \vec{B} \Rightarrow \lambda = \lambda(r)$

5.) Force-free fields only have $J_{\parallel}$, which can lead current-driven intabilies. The standard form of the Enegy integral δW has two destabilizing terms: one related to curvature $(\nabla P, \vec{\kappa})$ and one related to $\vec{J}_{\parallel}$.