2010 I : Q 5B   Math

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = 0 \quad \Rightarrow \quad y'' - xy' + 2y = 0$$

(A.) $x \to 0$ is an ordinary point so use power series method

$$y = \sum a_n (x-0)^n, \quad y' = \sum a_n n x^{n-1}, \quad y'' = \sum a_n n(n-1) x^{n-2}$$

$$\sum_0 a_n n(n-1) x^{n-2} + \sum_0 a_n n x^n + 2a_n x^n = 0$$

$$\sum a_{n+2}(n+2)(n+1) x^n + a_n[n+2] x^n = 0$$

$$\Rightarrow \quad a_{n+2} = -\frac{a_n}{n+1}$$

Two solutions: $y = \sum_{\text{odd}} a_n x^n$ and $y = \sum_{\text{even}} a_n x^n$.

(B.) $x \to \infty$

let $x = \frac{1}{t}$ $\quad \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} \quad \frac{dt}{dx} = -\frac{1}{x^2} = -t^2 \quad \frac{d}{dx}\frac{dy}{dt} = \frac{d}{dt}\frac{dt}{dx}\frac{dy}{dt}$

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2 + \frac{dy}{dt}\frac{d^2t}{dx^2} \quad \frac{d^2t}{dx^2} = \frac{2}{x^3} = 2t^3$$

$$\Rightarrow \ddot{y}t^4 + \dot{y}(2t^3) + \frac{1}{t}(-t^2)\dot{y} + 2y = 0$$

$$\ddot{y} + \frac{2}{t}\dot{y} - \frac{1}{t^3}\dot{y} + \frac{2}{t^4}y = 0 \quad \Rightarrow \text{irregular singular point}$$

Try $y = e^{S(x)}$ $\quad \Rightarrow \quad S'' + (S')^2 + xS' + 2 \sim 0$

$$(S')^2 + xS' + 2 \sim 0 \qquad S'(S' + x) \sim -2$$

$\Rightarrow$ 1) $\{(S')^2, xS'\} \gg 2$: $\underline{S' = -x}$ ✓

2) $\{(S')^2, 2\} \gg xS'$: $S' = \pm i\sqrt{2}$ ✗

3) $\{xS', 2\} \gg (S')^2$: $\underline{S' = -\frac{2}{x}}$ ✓

Using 1): $S' \sim -x \Rightarrow S = -\frac{1}{2}x^2$ let $S(x) = -\frac{x^2}{2} + g(x)$

$$-1 + g'' + (-x+g')^2 + x(-x+g') + 2 \sim 0 \qquad g' \ll x$$

$$\cancel{g''} + \cancel{x^2} - 2xg' + \cancel{(g')^2} - \cancel{x^2} + xg' + 1 \sim 0 \qquad g'' \ll 1$$

$$1 \sim xg', \quad g' \sim \frac{1}{x} \Rightarrow g \sim \ln(x) \Rightarrow \boxed{y_1(x) \sim xe^{-x^2/2}}$$

Using 3): $S' \sim -\frac{2}{x} \Rightarrow S = -2\ln(x) \Rightarrow \boxed{y_2(x) \sim xe^{-2}}$

??

c.) $y(x) = \int_C e^{xt} f(t)\,dt \qquad y'' + xy' + 2y = 0$

$$\int t^2 e^{xt} f(t)\,dt + x\int te^{xt} f(t)\,dt + 2\int e^{xt} f(t) = 0$$

$u = tf \quad v = e^{xt}$

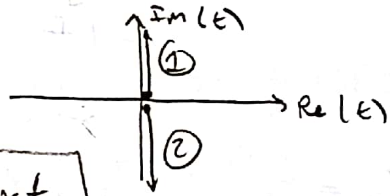
$du = (f + tf')du \quad dv = xe^{xt}dt$

$$\int (t^2+2) f(t) e^{xt}\,dt + \left[t f(t) e^{xt}\right]_C - \int \left(f(t) + tf'(t)\right)e^{xt}\,dt = 0$$

$$\Rightarrow (t^2+2)f = f + tf' \qquad f'/f = t + \frac{1}{t}$$

$$\int \frac{1}{f}\,df = \int t + \frac{1}{t}\,dt \Rightarrow \ln f = \frac{1}{2}t^2 + \ln t \Rightarrow f(t) = te^{\frac{1}{2}t^2}$$

$$\Rightarrow \left[t^2 e^{\frac{1}{2}t^2} e^{xt}\right]_C = 0 \Rightarrow$$



$$\boxed{y_1 = \int_0^{i\infty} dt\, te^{\frac{1}{2}t^2 + xt} \qquad y_2 = \int_0^{-i\infty} dt\, te^{\frac{1}{2}t^2 + xt}}$$

d.) $y(0) = 0, \quad y'(0) = 1$

Let $y(x) = A\int_0^{i\infty} dt\, te^{\frac{1}{2}t^2 + xt} - B\int_0^{-i\infty} dt\, te^{\frac{1}{2}t^2 + xt}$

$$y(0) = A\int_0^{i\infty} dt\, te^{\frac{1}{2}t^2} + B\int_{-i\infty}^{0} dt\, te^{\frac{1}{2}t^2} \qquad u = \frac{1}{2}t^2,\ du = t\,dt$$

$$= A\int du\, e^u + B\int du\, e^u = Ae^{\frac{1}{2}t^2}\Big|_0^{i\infty} + Be^{\frac{1}{2}t^2}\Big|_{-i\infty}^{0}$$

$y(0) = A + B = 0 \Rightarrow \underline{A = -B}$

$$y'(x) = A\int_0^{i\infty} dt \; t^2 e^{\frac{1}{2}t^2 + xt} - A\int_{-i\infty}^{0} dt \; t^2 e^{\frac{1}{2}t^2 + xt}$$

$$y'(0) = A\left[\int_0^{i\infty} dt \; t^2 e^{\frac{1}{2}t^2} + \int_{-i\infty}^{0} t^2 e^{\frac{1}{2}t^2} dt\right]$$

$$= A\left[\sqrt{\frac{\pi}{2}} + \sqrt{\frac{\pi}{2}}\right] = 1 \Rightarrow \underline{A = \frac{1}{\sqrt{2\pi}}}$$

$$\text{so } \boxed{y(x) = \frac{1}{\sqrt{2\pi}}\int_0^{i\infty} dt\, t e^{\frac{1}{2}t^2 + xt} - \frac{1}{\sqrt{2\pi}}\int_{-i\infty}^{0} dt\, t e^{\frac{1}{2}t^2 + xt}}$$

as $x \to \infty$ use Laplace's method:

$\phi(x,t) = \frac{1}{2}t^2 + xt \qquad \phi'(x,t) = t + x \qquad \phi''(t,x) = 1$

$\Rightarrow$ movable maxima at $t = -x$

let $s = -t/x$ then $I(x) = \int s x^2 e^{x^2\left(\frac{s^2}{2} - s\right)} ds \qquad t = -sx, \; dt = -x\,ds$

$\phi(s) = \frac{s^2}{2} - s \qquad \phi'(s) = s - 1 \qquad \phi''(s) = 1$

$$p = 2 \Rightarrow I(x) \sim \frac{\sqrt{2\pi}}{\sqrt{-x\phi''(c)}} f(c) e^{x\phi(c)} \qquad \underline{x \to x^2}$$

$$I(x) \sim \frac{\sqrt{2\pi}}{\sqrt{x^2}} x^2 e^{x^2\left(-\frac{1}{2}\right)}$$

$$\Rightarrow \boxed{y(x) \sim x e^{-x^2/2} \quad \text{as } x \to \infty}$$