2011 II : 1B MHD

a.) $m_e\left(\frac{\partial}{\partial t}+\vec{v}_e\cdot\nabla\right)\vec{v}_e = -e(\vec{E}+\vec{v}_e\times\vec{B}) - \frac{1}{n_e}\nabla P_e - \frac{1}{n_e}\vec{R}_e$

$m_i\left(\frac{\partial}{\partial t}+\vec{v}_i\cdot\nabla\right)\vec{v}_i = e(\vec{E}+\vec{v}_i\times\vec{B}) - \frac{1}{n_i}\nabla P_i - \frac{1}{n_i}\vec{R}_i$

NOTE that we also have

$\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e v) = 0, \quad \frac{\partial n_i}{\partial t} + \nabla\cdot(n_i v) = 0 \quad$ and $\quad \frac{d}{dt}\left(\frac{P}{n^\gamma}\right) = 0$

b.) Adding the two momentum equations gives: $\left(\sum_s \vec{R}_s = 0\right)$

$$m_e n_e \frac{dv_e}{dt} + m_i n_i \frac{dv_i}{dt} = en_i(\vec{E}+\vec{v}_i\times B) - en_i(\vec{E}+\vec{v}_e\times\vec{B}) - \nabla P_e - \nabla P_i$$

use quasineutrality: $en_i - en_e \approx 0$

neglect electron mass: $\rho = m_i n_i + m_e n_e \approx m_i n_i$

define $V \equiv \frac{\rho v_e + \rho v_i}{\rho_e + \rho_i} \approx V_i$, $\vec{J} \equiv en_i\vec{v}_i - en_e\vec{v}_e \approx en_i\vec{v}_i$

$\Rightarrow \rho\frac{dv}{dt} = \vec{J}\times\vec{B} - \nabla P_T \qquad P_T = P_i + P_e$

we also have $\rho\frac{\partial v}{\partial t} = en(\vec{E}+\vec{v}\times\vec{B}) - \nabla P_i - R_i$ from the ion equation. Combining yields

$$\vec{E}+\vec{v}\times\vec{B} = \frac{1}{en}(\vec{J}\times\vec{B}) - \frac{1}{en}\nabla P_e - \frac{1}{en}\vec{R}_i$$

Letting $\vec{R}_i = -en\eta\vec{J}$ yields

$$\boxed{\vec{E}+\vec{v}\times\vec{B} = \frac{1}{en}(\vec{J}\times\vec{B}) - \frac{1}{en}\nabla P_e + \eta\vec{J}}$$

c.) Note that we have already dropped the electron inertia term. We get rid of the Hall term $\frac{1}{en}(\vec{J}\times\vec{B})$ by assuming that the plasma is highly magnetized ($\rho_i \ll L$). This condition also lets us drop the electron pressure term $-\frac{1}{en}\nabla P$, leaving us with $\vec{E}+\vec{v}\times\vec{B}=\eta\vec{J}$. If the plasma is infinitly conductive $\left(\frac{V_{E\times B}}{V_{th}}\sim 1\right)$ then we can also drop the $\eta\vec{J}$ term to get $\vec{E}+\vec{v}\times\vec{B}=0$ (ideal MHD)

d.) If these conditions aren't satisfied in fusion plasmas they can usually still be applied since the model tends to breakdown in regions that are not important. For example, MHD works well for tokamaks, but not for FRCs or the scrape off layer.

I'm not sure what the second part of this question is after... The MHD assumptions are

1) highly collisonal $\lambda_{mfp} \ll L$
2) non-relativistic $v_p \ll c$
3) quasi-neutrality $\lambda_D \ll L$
4) highly magnetized $\rho_i \ll L$
5) infinitly condutive $V_{E\times B}/v_{th} \sim 1$