2011 II : 3A General

a.) $\frac{dn}{dt} + \nabla\cdot(n\vec{v}) = 0$ $\quad$ $m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) - \cancel{\frac{1}{n}\nabla P}^{\,0}$

$n \to n_0 + \tilde{n}$, $\vec{v} \to \tilde{v}$

$\Rightarrow \partial_t \tilde{n} + \partial_x (n_0 \tilde{v}) = 0$ $\quad$ $m\,\partial_t \tilde{v} = q(\tilde{E} + \tilde{v}\times\cancel{\vec{B}_0}^{\,0})$

$-i\omega\tilde{n} + ikn_0\tilde{v} = 0$ $\quad$ $-m_e i\omega\tilde{v} = q\tilde{E}$

Also use $\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ $\Rightarrow ik\tilde{E} = \frac{q\tilde{n}}{\epsilon_0}$ then

$-m_e i\omega\tilde{v} = q^2\frac{\tilde{n}}{ik\epsilon_0}$ $\Rightarrow \tilde{n} = m\omega k\epsilon_0/q^2\ \tilde{v}$

$\Rightarrow \omega\left(\frac{m\omega \cancel{k}\epsilon_0}{q^2}\right) = \cancel{k}n_0$ $\Rightarrow \omega^2 = \frac{q^2 n_0}{\epsilon_0 m}$ $\Rightarrow \boxed{\omega^2 = \omega_p}$

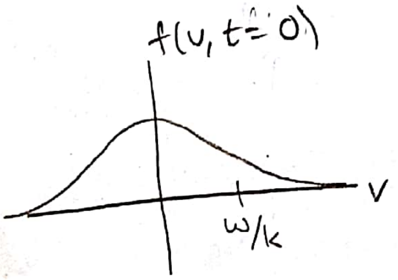
b.) $E = E_0\cos(kx - \omega_p t) \Rightarrow \nabla\cdot\vec{E} = kE_0\cos(kx - \omega_p t)$

$\Rightarrow \boxed{\tilde{n} = \frac{\epsilon_0 k}{q}E_0\cos(kx)}$ $\quad$ From $\nabla\cdot\vec{E} = \frac{qn}{\epsilon_0}$

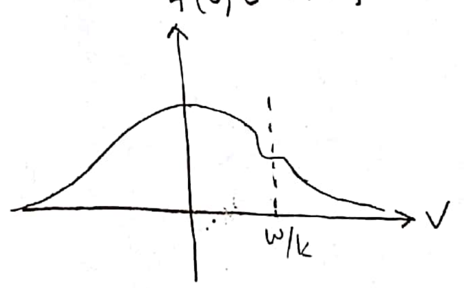
and $m\dot{v} = qE$, $\dot{v} = \frac{q}{m}E$ $\quad$ $\int\cos(kx - \omega t)\,dt = -\frac{1}{\omega}\sin(kx - \omega t)$

$\Rightarrow \boxed{\tilde{v} = -\frac{q}{m\omega_p}E_0\sin(kx)}$

c.)

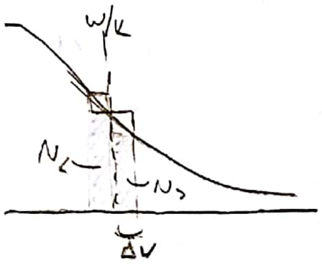


$\rightarrow$



d.) For $E(x, t\to\infty)$ to remain finite, the wave must have more energy than the particles take from it.

First Find energy transfered to particles:



$$U = \int_{w/k-\Delta v}^{w/k+\Delta v} dv \quad m v^2 \Delta f_0(v) \qquad \Delta f_0(v) \approx \Delta v f_0'\left(\frac{\omega}{k}\right) + \frac{(\Delta v)^2}{2} f_0''\left(\frac{\omega}{k}\right)$$

$$\Rightarrow U \approx m \int_{\frac{\omega}{k}-\Delta v}^{\frac{\omega}{k}+\Delta v} dv \quad v^2 \Delta v \left|f_0'\left(\frac{\omega}{k}\right)\right|$$

Note that $\Delta v = v - \frac{\omega}{k}$ $\Longrightarrow$ $U \approx m \int_{-v_{tr}}^{v_{tr}} d(\Delta v) \quad \Delta v \left(\Delta v + \frac{\omega}{k}\right)^2 \left|f_0'\left(\frac{\omega}{k}\right)\right|$

$$\Rightarrow U \approx m \frac{2\omega}{k}\left|f_0'\left(\frac{\omega}{k}\right)\right|\left(\frac{\Delta v^3}{3}\right)\Bigg|_{-v_{tr}}^{v_{tr}} = \frac{4}{3} m \frac{\omega}{k}\left|f_0'\left(\frac{\omega}{k}\right)\right| v_{tr}^3$$

The energy in the wave is $|E|^2/8\pi$

$$\Rightarrow \frac{|E|^2}{8\pi} > \frac{4}{3} m \frac{\omega}{k}\left|f_0'\left(\frac{\omega}{k}\right)\right| v_{tr}^3$$

Using $v_{tr}^2 = \frac{eE}{mk}$ $\Rightarrow$ $|E|^2 = \frac{v_{tr}^4 m^2 k^2}{e^2}$

$$\Rightarrow \quad v_{tr} > c \frac{e^2}{m} \frac{\omega}{k^3}\left|f_0'\left(\frac{\omega}{k}\right)\right|$$

The wave we are interested in has $\omega = \omega_p \equiv \frac{q^2 n_0}{\varepsilon_0 m}$

$$\Rightarrow \boxed{v_{tr} > c \, \frac{\omega_p^3}{k^3}\left|f_0'\left(\frac{\omega}{k}\right)\right|}$$