2011 I: 1B Math

$$\frac{d^2\psi}{dx^2} + Q(x)\psi = 0 \qquad \text{consider } \psi = e^{S(x)}$$

then $S'' + (S')^2 + Q = 0$

assume $(S')^2 >> S'' \Rightarrow S'(x) = \pm i\sqrt{Q(x)} + g'(x)$

plug back in: $g'' \pm i \frac{1}{2}\frac{1}{\sqrt{Q}}\frac{dQ}{dx} + (g')^2 \pm 2ig'\sqrt{Q} - \cancel{Q} + \cancel{Q} = 0$

NOTE: $g'(x) << \sqrt{Q(x)}$

$g''(x) << \frac{1}{2}\frac{1}{\sqrt{Q}}\frac{dQ}{dx}$

so keeping only largest terms yields

$$\pm 2\cancel{i}g'\sqrt{Q} \pm \cancel{i}\frac{1}{2\sqrt{Q}}Q' = 0 \quad \Rightarrow g' = -\frac{1}{4}\frac{Q'}{Q}$$

$$S(x) = \pm i\int\sqrt{Q(x)}\,dx \qquad g(x) = -\frac{1}{4}\ln(Q(x))$$

so $\psi(x) \sim \exp\left[\pm i\int\sqrt{Q(x)}\,dx - \frac{1}{4}\ln(Q(x))\right]$

$$\boxed{\psi(x) \sim \frac{1}{Q^{1/4}}\exp\left(\pm i\int dx\,\sqrt{Q(x)}\right)}$$

$Q = 0$ blows up, so the solution clearly fails

Also, need $S'' << (S')^2 \Rightarrow \frac{S''}{(S')^2} << 1$

$$\boxed{Q^{-3/2}\frac{dQ}{dx} << 1}$$