2011 I: Q2 Waves

a.) In the lab frame, we know that $\phi \sim e^{i(kx-\omega t)}$, so to get $\phi(x,t)$ static we need $\frac{x}{t} - \frac{\omega}{k} = 0$. So a frame K' moving at the phase velocity $v_p = \omega/k$ has static $\phi(x,t)$.

A local particle has energy $\frac{1}{2}mw^2 = \frac{1}{2}mw_0^2 + e\phi(x)$

$$\Rightarrow \boxed{w = \sqrt{w_0^2 + \frac{2e}{m}\phi(x)}}$$

b.) $|w|\,dn_e = |w_0|\, n_0 f_w(w_0)\, dw_0$ Poisson's Eq: $\nabla^2\phi = -4\pi\rho_q = -4\pi e(n_i - n_e)$

$$\int |w|\,dn_e = \int |w_0| n_0 f_w(w_0)\,dw_0$$

$$\frac{n_e}{n_0} = \int \frac{|w_0|}{|w|} f_w(w_0)\,dw_0 = \int \frac{|w_0|}{\sqrt{w_0^2 + \frac{2e}{m}\phi}} f_w(w_0)\,dw_0$$

neglect ion motion $(n_i = n_0)$

$$\frac{n_e}{n_i} = \int \left(1 + \frac{2e}{mw_0^2}\phi\right)^{-1/2} f_w(w_0)\,dw_0$$

$$\text{so } \nabla^2\phi = -4\pi e n_i \left(1 - \underbrace{\int \left(1 + \frac{2e}{mw_0^2}\phi\right)^{-1/2} f_w(w_0)\,dw_0}_{\text{Taylor expand}}\right)$$

$$-k^2\phi \approx -4\pi e n_i\left(\cancel{1} - \cancel{\int f_w(w_0)\,dw_0} - \int \phi \frac{e}{mw_0^2} f_w(w_0)\,dw_0\right)$$

$$+k^2 \approx \underbrace{+4\pi e n_i \frac{e}{m}}_{\omega_p^2} \int \frac{f_w(w_0)}{w_0^2}\,dw_0$$

change $w_0 \Rightarrow v - \frac{\omega}{k}$ to get $dw_0 = dv$

$$0 = -k^2 + \omega_p^2 \int_{-\infty}^{\infty} \frac{f_0(v)}{(v-\omega/k)^2}\,dv$$

$$\Rightarrow \boxed{1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{f_0(v)}{(v-\omega/k)^2}\,dv = 0}$$

c.) $1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{f_0(v)}{(\omega/k)^2(1-kv/\omega)^2}\, dv = 0$ $\qquad 1 + \frac{kv}{\omega}\frac{-2}{(1-0)^3}(-1)$

Taking $\frac{kv}{\omega} << 1$ gives $1 \approx \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty}\frac{k^2}{\omega^2} f_0(v)\left(1+\frac{2kv}{\omega}\right)dv$

$\left(\frac{v}{v_p} << 1\right)$

$$\Rightarrow \frac{k^2}{\omega_p^2} = \frac{k^2}{\omega^2}\overbrace{\cancel{\int_{-\infty}^{\infty} f_0(v)\,dv}}^{1} + \frac{k^2}{\omega^2}\frac{2k}{\omega}\int_{-\infty}^{\infty} v f_0(v)\, dv$$

Assuming a Maxwellian $f_0(v) = \frac{1}{\sqrt{2\pi}}\frac{1}{v_{th}} e^{-v^2/2v_{th}^2}$ gives

$$\frac{k^2}{\omega_p^2} = \frac{k^2}{\omega^2}\left(1 + \frac{2k}{\omega}\overbrace{\cancel{\int_{-\infty}^{\infty}\frac{v}{\sqrt{2\pi}}\frac{1}{v_{th}} e^{-v^2/2v_{th}^2}\,dv}}^{0}\right)$$

Need to keep one more term for thermal corections.

$$\frac{1}{(1-\frac{kv}{\omega})^2} \approx 1 + \frac{2kv}{\omega} + \left(\frac{kv}{\omega}\right)^2\frac{6}{(1-0)^2}\frac{1}{2!} = 1 + \frac{2kv}{\omega} + 3\left(\frac{kv^2}{\omega^2}\right)$$

$$\frac{\cancel{k^2}}{\omega_p^2} = \frac{\cancel{k^2}}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\,v_{th}} v^2 e^{-v^2/2v_{th}^2}\,dv\right)$$

$$= \frac{1}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2\cancel{\sqrt{2\pi}\,v_{th}}}\left(\cancel{\sqrt{2\pi}}\, v_{th}^{\cancel{3}}\right)\right)$$

$$\omega^2 = \omega_p^2\left(1 + 3\frac{k^2 v_{th}^2}{\omega^2}\right)$$

$$\omega^4 - \omega_p^2\omega^2 - 3\omega_p^2k^2v_{th}^2 = 0$$

$$\omega^2 = \frac{\omega_p^2 \pm \sqrt{\omega_p^4 + 4\cdot 3\omega_p^2k^2v_{th}^2}}{2} = \frac{\omega_p^2 + \omega_p^2\sqrt{1+\frac{12k^2v_{th}^2}{\omega_p^2}}}{2}$$

$$\Rightarrow \omega^2 \approx \frac{\omega_p^2}{2} + \frac{\omega_p^2}{2}\left(1+\frac{12k^2v_{th}^2}{2\omega_p^2}\right) \Rightarrow \boxed{\omega^2 = \omega_p^2 + 3k^2v_{th}^2}$$