2011 I : QSB Math

$y'' - xy' - y = 0$

a.) Take $x \to +\infty$

$t = \frac{1}{x}$ $\quad \frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = -\frac{1}{x^2}\frac{dy}{dt} = -t^2\frac{dy}{dt}$

$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\left(-\frac{1}{x^2}\right)\frac{dy}{dt} + -\frac{1}{x^2}\left(\frac{dt}{dx}\frac{d}{dt}\frac{dy}{dt}\right) = 2t^3\dot{y} + t^4\ddot{y}$

$t^4\ddot{y} + 2t^3\dot{y} + t^3\dot{y} - y = 0$ has irregular singular point at $t=0$.

Try $y(x) = e^{S(x)}$

$S'' + (S')^2 - xS' - 1 = 0$

① $\{(S')^2, xS'\} >> \{S'', 1\} \Rightarrow S'(S'-x) = 0 \Rightarrow \underline{\underline{S' = x}}$

then $S'' = 1 << x, \quad 1 << x$ ✓

$\{(S')^2, 1\} >> \{S'', xS'\} \Rightarrow S' = \pm 1$ ✗

② $\{xS', 1\} >> \{S'', (S')^2\} \Rightarrow S' = -\frac{1}{x}$ ✓

then $S'' = (-S')^2 = \frac{1}{x^2} << 1$.

For solution ①: $S' = x + g'$ yields

$\cancel{1} + \cancel{g''} + \cancel{x^2} + 2xg' + \cancel{(g')^2} - \cancel{x^2} - xg' - \cancel{1} = 0$

$g'' << 1$ $\quad (g')^2 << xg'$ $\quad \Rightarrow g' = 0$

① $\Rightarrow \boxed{y_1(x) \sim e^{\frac{1}{2}x^2 + c}}$

For solution ② $S'(x) = \ln(x)$ is already a log term so

② $\Rightarrow \boxed{y_2(x) \sim x}$

b.) As $x \to 0$ $y'' - xy' - y = 0$ has $x=0$ as a regular

so try $y(x) = \sum a_n x^n$ $y' = \sum a_n n x^{n-1}$ $y'' = \sum a_n n(n-1) x^{n-2}$

$$\Rightarrow \sum a_n n(n-1) x^{n-2} - \sum a_n n x^n - \sum a_n x^n = 0$$

$n \to n+2$

$$\sum_{n=-2}^{\infty} a_{n+2}(n+2)(n+1) x^n - \sum_{n=0}^{\infty} a_n (n+1) x^n = 0$$

$$a_{n+2}(n+2)\cancel{(n+1)} - a_n \cancel{(n+1)} = 0$$

$$\Rightarrow \boxed{a_{n+2} = \frac{a_n}{n+2}} \quad n \geq 0$$

Note: a_0, a_1 free so use to define two solutions:

$$y_1(x) \sim \sum_{n=0}^{\infty} a_n x^n \quad \text{with } n_0 = 0 \qquad a_{2(n+1)} = \frac{a_{2n}}{2(n+1)}$$

$$y_2(x) \sim \sum_{n=0}^{\infty} a_n x^n \quad \text{with } n_0 = 0 \qquad a_{2n+3} = \frac{a_{2n+1}}{2n+3}$$

$\Rightarrow$

$$y_1(x) \sim \sum_{n=0}^{\infty} a_{2n} x^{2n} \quad \text{where } a_{2n} = \frac{a_0}{2^n (n+1)!}$$

$$y_2(x) \sim \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} \quad \text{where } a_{2n+1} =$$

c.) $y(x)$ with $y(x \to +\infty) \to 0$: Use integral representation

$$y(x) = \int_C f(x) e^{xt} \, dt \quad \Rightarrow y' = \int_C dt \, t e^{xt}$$

$$y'' = \int_C dt \, t^2 e^{xt}$$

$$\int_C dt \, t^2 f e^{xt} - \int_C dt \, x t f e^{xt} - \int_C dt \, f e^{xt} = 0$$

$u = tf$ $\quad v = e^{xt}$

$du = (f + t f')dt$ $\quad dv = x e^{xt} dt$

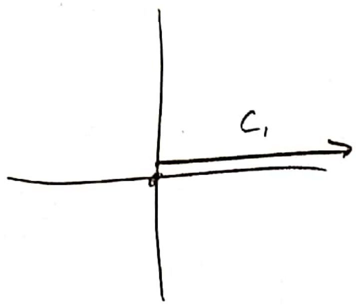
$$\int_C dt \, t^2 f e^{xt} - \left[t f e^{xt} \right]_C + \int_C dt \, (f + t f') e^{xt} dt - \int_C dt \, f e^{xt} = 0$$

so $t^{\cancel{2}} f(t) + \cancel{f(t)} + \cancel{t} f'(t) - \cancel{f(t)} = 0$

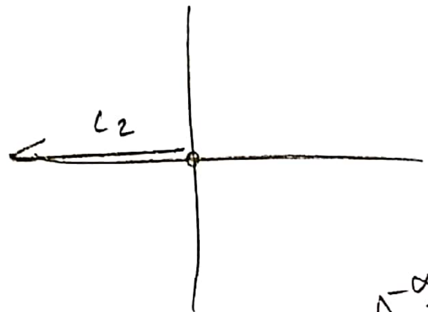
$$\frac{f'(t)}{f(t)} = -t \qquad f(t) = e^{-\frac{1}{2}t^2}$$

so $t e^{-\frac{1}{2}t^2} e^{xt} \Big|_C = 0$ as well, giving us these contours:

$-t^2 \to \infty$ at $\pm\infty$



and



so $y_1(x) = \int_0^{\infty} e^{-\frac{1}{2}t^2} e^{xt}\, dt$ and $y_2(x) = \int_0^{-\infty} e^{-\frac{1}{2}t^2} e^{xt}\, dt$

Choose $y_2(x) = \int_0^{\infty} e^{-\frac{1}{2}t^2} e^{-xt}\, dt$ so that $y(x \to +\infty) \to 0$

as $x \to 0$, $y_2(x) \sim \int_0^{\infty} e^{-\frac{1}{2}t^2}\, dt = \frac{\sqrt{2\pi}}{2} \Rightarrow \boxed{y(0) \sim \sqrt{\frac{\pi}{2}}}$

as $x \to \infty$, use Laplace's method.

$$\phi(x,t) = -\frac{1}{2}t^2 - xt \qquad \phi'(x,t) = -t - x \qquad \phi''(x,t) = -1$$

$\Rightarrow$ movable maxima at $x = -t$

let $s = -t/x$, $t = -sx$, $dt = -x\,ds$ then

$$I(x) = \int_0^{\infty} ds\, -x\, e^{x^2(-\frac{s^2}{2} + s)}$$

$$\phi(s) = s - \frac{1}{2}s^2 \qquad \phi'(s) = 1 - s \qquad \phi''(s) = -1$$

$p = 2$, $c = 1 \Rightarrow I(x) \sim \frac{\sqrt{2\pi}}{\sqrt{-x^2\phi''(c)}} f(c) e^{x^2\phi(c)}$

$$\boxed{y(x \to \infty) \sim -\sqrt{2\pi}\, e^{x^2/2}}$$