2012 II : Q3 Math

$$xy'' + y' - xy = 0 \quad \Rightarrow \quad y'' + \frac{1}{x}y' - y = 0$$

a.) $x \to +\infty$, let $t \to \frac{1}{x}$ then $\frac{dy}{dx} = \frac{dt}{dx}\frac{dy}{dt} = -t^2 \dot{y}$ $\quad \frac{dt}{dx} = -\frac{1}{x^2}$

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\left(\frac{dt}{dx}\right)\frac{dy}{dt} + \frac{dt}{dx}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -2t^3\dot{y} + \left(\frac{dt}{dx}\right)^2 \ddot{y} = -2t^3\dot{y} + t^4\ddot{y}$$

$$\Rightarrow \quad t^4\ddot{y} - 2t^3\dot{y} + t^3\dot{y} - y = 0$$

$$\ddot{y} - \frac{1}{t}\dot{y} - \frac{1}{t^4}y = 0 \qquad t = 0 \text{ is } \underline{\text{irregular singular point.}}$$

Try $y(x) \sim e^{S(x)}$ $\Rightarrow$ $S'' + (S')^2 + \frac{1}{x}S' - 1 = 0$

$$\{(S')^2, 1\} \gg \{S'', \tfrac{1}{x}S'\} \rightarrow S' = \pm 1 \gg \{0, \tfrac{1}{x}\} \checkmark$$

then let $S' = \pm 1 + g'$ with $g' \ll 1$

$$x(g'' + \cancel{1} \pm 2g' + (g')^2) \pm 1 + g' - \cancel{x} = 0$$

$$xg'' + x(g')^2 + (\pm 2x + 1)g' \pm 1 \sim 0$$

Try dominant balances:

$\cancel{x(g')^2 \pm 2xg' \sim 0} \rightarrow g' \sim \mp 2$ violates $g' \ll 1$

$\pm 2xg' \pm 1 \sim 0 \rightarrow g' \sim -\frac{1}{2x} \Rightarrow g \sim -\frac{1}{2}\ln(x)$

$$\Rightarrow \boxed{y(x) = \frac{1}{\sqrt{x}}e^{\pm x}}$$

b.) $x \to 0$ $\rightarrow$ regular singular point.

Try $y(x) \sim x^r \sum_n a_n x^n$

$$\sum a_n\Big[(r+n)(r+n-1)x^{r+n-1} + (r+n)x^{r+n-1} + \underbrace{x^{r+n+1}}_{\hookrightarrow\, n \to n-2}\Big] = 0$$

$$\sum_{n=0}^{\infty} a_n\left[(r+n)(r+n-1)+(r+n)\right]x^{r+n-1} + \sum_{n=2}^{\infty} a_{n-2}\, x^{r+n-1} = 0$$

$[n=0]$: $a_0(r(r-1)+r)=0 \Rightarrow a_0 r^2 = 0 \Rightarrow$ double root!

$[n=1]$: $a_1((r+1)r+(r+1))=0 \Rightarrow a_1(r+1)^2=0 \Rightarrow a_1=0$

$[n\geq 2]$: $a_n = (r+n)^{-2} a_{n-2} \Rightarrow a_n = \dfrac{a_{n-2}}{n^2}$

so $y_1(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow \boxed{y_1(x) \sim 1 + \frac{1}{4}x^2}$

$$y_2(x) = y_1(x)\ln x + x^0 \sum_{n=0}^{\infty} \left.\frac{\partial a_n(r)}{\partial r}\right|_{r=0} x^n \qquad \frac{\partial a_n}{\partial r} = -2\frac{a_{n-2}}{(r+n)^3}$$

$$= y_1(x)\ln(x) + \sum_{n=2}^{\infty} b_n x^n \quad \text{where} \quad b_n = -2\frac{a_{n-2}}{n^3}$$

$$y_2(x) \sim \ln(x) + \frac{x^2}{4}\ln(x) - \frac{2}{8}x^2 \Rightarrow \boxed{y_2(x) \sim \ln(x)}$$

(c.) Let $y(x) = \int_C dt\, f(t)e^{xt}$

$$x\int_C dt\, t^2 f(t)e^{xt} + \int_C dt\, t f(t)e^{xt} - \int_C dt\, x f(t)e^{xt} = 0$$

$u = t^2 f(t)$ $\quad v = e^{xt}$

$2tf + t^2 f'$ $\quad dv = xe^{xt}\,dt$

$$\int_C dt\left(-\left[2tf(t)+t^2f'(t)\right] + tf(t) + f'(t)\right)e^{xt}$$

$$+\left[t^2 f(t)e^{xt} - f(t)e^{xt}\right]_C = 0$$

$$\Rightarrow -tf(t) + (1-t^2)f'(t) = 0$$

$u = t^2+1$

$du = 2t\,dt$

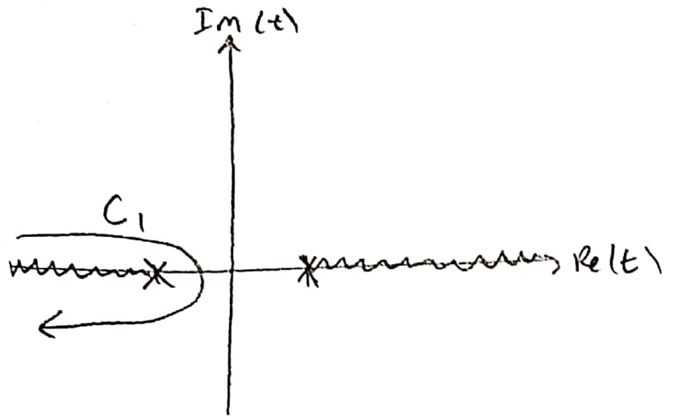
$$\frac{f'(t)}{f(t)} = -\frac{t}{t^2-1} \Rightarrow f(t) = \frac{1}{\sqrt{t^2-1}}$$

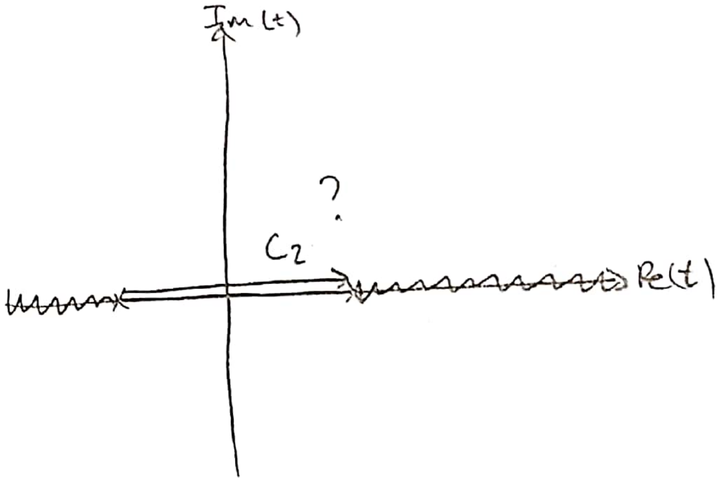
$$\int \frac{t}{t^2-1}dt = \frac{1}{2}\int\frac{1}{u}du = \frac{1}{2}\ln(u)$$

To find the contours:

$$\left[t^2 \frac{1}{\sqrt{t^2-1}} e^{xt} - \frac{1}{\sqrt{t^2-1}} e^{xt}\right]_C = 0$$

$$\frac{t^2-1}{\sqrt{t^2-1}} e^{xt}\Big|_C = 0 \quad \Rightarrow \quad \sqrt{t^2-1}\, e^{xt}\Big|_C = 0$$




so $\displaystyle y_1(x) = \int_{-\infty}^{(-1+)} dt \, \frac{e^{xt}}{\sqrt{t^2-1}}$ $\qquad$ $\displaystyle y_2(x) = \int_{-1}^{1} dt \, \frac{e^{xt}}{\sqrt{t^2-1}}$