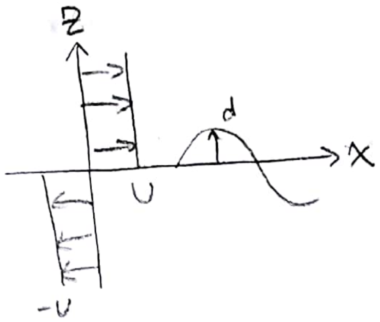
2012 II : Q4 MHD



incompressible, ideal

$$\vec{U}_0 = \begin{cases} (U,0,0) & z>0 \\ (-U,0,0) & z<0 \end{cases}$$

1.) $\vec{u} = \vec{U}_0 + \vec{v}(x,z)$ where $\vec{v}(x,z) = \nabla\phi$

incompressible: $\nabla\cdot\vec{v} = 0$ so $\nabla^2\phi = 0$

for $z>0$, $\nabla^2 \phi_1(z) e^{ik(x-ct)} = 0$

$$\Rightarrow -k^2\phi_1(z)e^{ik(x-ct)} + \phi_1''(z)e^{ik(x-ct)} = 0$$

$$\phi_1''(z) = k^2\phi_1(z)$$

$$\phi_1(z) = C_1 e^{kz} + C_2 e^{-kz} \qquad \phi(\infty) = 0$$

$$\boxed{\begin{aligned}\phi_1(z) &= Ae^{-kz} \\ \text{similarily} \quad \phi_2(z) &= Be^{kz}\end{aligned}}$$

2.) Suppose $\xi = de^{ik(x-ct)}$, $v_z = \frac{D\xi}{Dt} = \frac{\partial\xi}{\partial t} + (\vec{u}\cdot\nabla)\xi = \frac{\partial}{\partial z}\phi$

for $z>0$, $d(-i\cancel{k}c)\cancel{e^{ik(x-ct)}} + ud(i\cancel{k})\cancel{e^{ik(x-ct)}} = A(-\cancel{k})e^{-kz}\cancel{e^{ik(x-ct)}}$ $\quad {}^{1,\,z>0}$

$$\boxed{\begin{aligned}A &= -id(u-c) \\ B &= -id(u+c)\end{aligned}}$$

3.) $p_1 \equiv p(z \to +0)$

In this case the momentum equation is $\rho\frac{D\vec{v}}{Dt} = \cancel{\vec{J}\times\vec{B}} - \nabla P$ $\quad \vec{B}=0$

$$\rho\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\nabla\phi = -\nabla p \qquad \text{assume } P \sim p(z)e^{ik(x-ct)}$$

$$z \to +0 \Rightarrow \rho\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)i\cancel{k}Ae^{ik(x-ct)} = -\cancel{ik}p_1e^{ik(x-ct)}$$

$$\rho(-ikc + iku)A = -p_1$$

similarily, $p_2 = -\rho B(-ikc - iku)$

$P_1 = P_2 \Rightarrow B(-ikc - iku) = A(-ikc + iku)$

$(kc + ku)(kc + ku) = (ku - kc)(kc - ku)$

$(kc)^2 + 2\cancel{(ku)(kc)} + (ku)^2 = -(ku)^2 - (kc)^2 + 2\cancel{(kc)(ku)}$

dispersion relation $\Rightarrow$ $\boxed{kc = \pm iku}$

This system is unstable if $\mathrm{Im}(kc = \omega) > 0$, which corresponds to $u \neq 0$.

4.) Impose $\vec{B}_0 = B_0 \hat{x}$

$\vec{E} + \vec{v} \times \vec{B} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\Rightarrow \nabla \times (\vec{v} \times \vec{B}) = +\frac{\partial \vec{B}}{\partial t}$

$\vec{v} = \vec{u}_0 + \vec{v}_1, \quad \vec{B} = \vec{B}_0 + \vec{B}_1$

$\cancelto{0}{\nabla \times (\vec{v}_0 \times \vec{B}_0)} + \nabla \times (\vec{v}_1 \times \vec{B}_0) + \nabla \times (\vec{v}_0 \times \vec{B}_1) = +\frac{\partial \vec{B}_1}{\partial t}$

$\nabla \cdot B = 0, \quad \nabla \cdot v = 0$

$\Rightarrow (\vec{B}_0 \cdot \nabla)\vec{v}_1 - \cancelto{0}{(\vec{v}_1 \cdot \nabla)\vec{B}_0} + \cancelto{0}{(\vec{B}_1 \cdot \nabla)\vec{v}_0} - (\vec{v}_0 \cdot \nabla)\vec{B}_1 = +\frac{\partial \vec{B}_1}{\partial t}$

$B_0\left(\frac{\partial}{\partial x}\right)^2 \phi - u \frac{\partial}{\partial x} B_1 = \frac{\partial}{\partial t} B_1$

$z > 0: \; B_0 (ik)^2 \phi_1(z) \cancel{e^{ik(x-ct)}} - B_x u (ik) \cancel{e^{ik(x-ct)}} = B_x(-ikc) \cancel{e^{ik(x-ct)}}$

$B_x(ikc + iku) = B_0(-k^2)\phi_1(z)$

$$B_x = -\frac{k^2}{iku - ikc} B_0 \phi_1(z) \qquad = k B_0 d$$

for $z < 0$:

$$B_x = \frac{k^2}{iku + ikc} B_0 \phi_2(z) \qquad = -k B_0 d$$

$$P_1 + \frac{B_x B_0}{\mu_0} = P_2 + \frac{B_x B_0}{\mu_0}$$

$$-\rho(-ikc + iku)(-i\alpha(u-c)) + \frac{B_0^2}{\mu_0}k\alpha = -\rho(-ikc - iku)(-i\alpha(u+c)) - \frac{B_0^2}{\mu_0}k\alpha$$

$$\frac{2B_0^2}{\mu_0} = \rho\left[(-ic + iu)(-iu + ic) - (-ic - iu)(-iu - ic)\right]$$

$$\frac{2B_0^2}{\mu_0} = -\rho\left[\underbrace{(-c+u)(-u+c)}_{2uc - u^2 - c^2} - \underbrace{(-c-u)(-u-c)}_{-u^2 - c^2 - 2uc}\right] \qquad V_A^2 = \frac{B_0^2}{\mu_0\rho}$$

$$\frac{\cancel{2}B_0^2}{\mu_0} = +\rho\left[+\cancel{2}u^2 + c^2\right] \Rightarrow \boxed{u^2 + c^2 = V_A^2}$$

Need $\mathrm{Im}(kc = \omega) > 0$ for instability

$$(kc)^2 = k^2(V_A^2 - u^2)$$

$$k^2(V_A^2 - u^2) \geq 0 \quad \text{if } V_A^2 \geq u^2 \Rightarrow \frac{B_0^2}{\mu_0\rho} \geq u^2$$

so $B_0^2 \geq \mu_0 \rho u^2$ will stabilize the instability

or, if we shift reference frames, $V_A \geq \frac{1}{2}|u_1 - u_2|$