2012 I : Q2 Waves

(a.) $\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla_x f + \vec{F}\cdot\nabla_p f = 0$ $\vec{F} = e\vec{E} = -e\nabla\phi$, $\vec{E} = E\hat{k}$

let $f = f_0 + f_1 \Rightarrow \partial_t f_1 + \vec{v}\cdot\nabla f_1 + q\vec{E}\cdot\nabla_p f_0 = 0$

$\Rightarrow -i\omega f_1 + ikvf_1 + e\vec{E}\frac{\partial f_0}{\partial p} = 0 \Rightarrow f_1 = \frac{ie\vec{E}}{\vec{k}\vec{v}-\omega}\cdot\frac{\partial f_0}{\partial \vec{p}}$

$\vec{J} = \int d^3p\, e\vec{v}f_1 = \int d^3p \frac{ie^2}{\vec{k}\cdot\vec{v}-\omega}\vec{v}\cdot\vec{E}\frac{\partial f_0}{\partial \vec{p}} \xrightarrow[\vec{E}=E\hat{k}]{\text{(longitudinal)}} = \frac{ie^2}{k}\int d^3p \frac{\frac{\vec{k}\cdot\vec{v}}{\omega}}{\frac{\vec{k}\cdot\vec{v}}{\omega}-1}\frac{\partial f_0}{\partial \vec{p}} E$

$\int_0^\pi \sin\theta\cos\theta\, d\theta = 0$

$= \frac{ie^2}{k}\int d^3p \left(\cancel{1} + \frac{1}{\vec{k}\cdot\vec{v}/\omega - 1}\right)\cdot\frac{\partial f_0}{\partial \vec{p}}\cdot E\hat{k} \Rightarrow \sigma = \frac{ie^2}{k}\omega\int d^3p \frac{\hat{k}}{\vec{k}\cdot\vec{v}-\omega}\cdot\frac{\partial f_0}{\partial \vec{p}}$

$\epsilon(\omega,k) = 1 + \frac{4\pi i\sigma}{\omega} \Rightarrow \epsilon = 1 - \frac{4\pi e^2}{k}\int d^3p \frac{1}{\vec{k}\cdot\vec{v}-\omega}\hat{k}\cdot\frac{\partial f_0}{\partial \vec{p}}$

let $\vec{k} = k\hat{z}$, $p_{\parallel} = p\cos\theta$

$$\epsilon(\omega,k) = 1 - \frac{4\pi e^2}{k}\int_0^{2\pi} d\phi_p \int_0^\infty p^2\, dp \int_0^\pi \sin\theta\, d\theta \frac{\hat{k}}{\vec{k}\cdot\vec{v}-\omega}\cdot\frac{\partial f_0}{\partial \vec{p}}$$

$\vec{k}\cdot\vec{v} = \vec{k}\cdot c\hat{p} = ck\cos\theta$ $\hat{z}\cdot\frac{\partial f_0}{\partial \vec{p}} = -\frac{c}{T}\cos\theta\, f_0(p)$

"ultrarelativistic" $v \approx c$

$$\Rightarrow \epsilon(\omega,k) = 1 + \frac{4\pi e^2}{kT}\int_0^\infty f_0(p)\,2\pi p^2\, dp \int_0^\pi \frac{c\cos\theta\ {}^{1/kc}}{ck\cos\theta - \omega/_{kc}}\sin\theta\, d\theta$$

let $u \equiv \frac{\omega}{kc}$

$$\Rightarrow \epsilon(\omega,k) = 1 + \frac{4\pi e^2}{k^2T}\int_0^\infty f_0(p)\,2\pi p^2\, dp \int_0^\pi \frac{\cos\theta}{\cos\theta - u}\sin\theta\, d\theta$$

NOTE: It's easier (faster) just to use

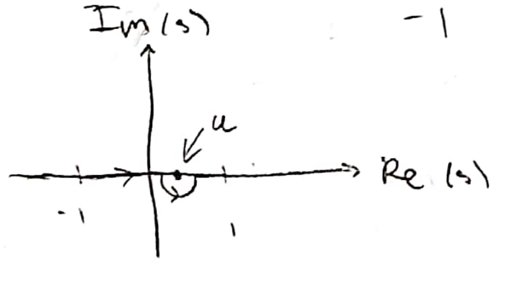
$\nabla\cdot\vec{E} = 4\pi q n_1 = 4\pi q\int f_1\, dp$

since the wave is electrostatic

(b.) $\displaystyle\int_0^{\pi} \frac{\cos\theta}{\cos\theta - u}\sin\theta \, d\theta \stackrel{?}{=} \int_{-1}^{1} \frac{s}{s-u}\, ds$ → divergent (need to use the Landau contour)

$\displaystyle\epsilon(\omega, k) = 1 + \frac{2\pi e^2 n_0}{k^2 T}\int_{-1}^{1} \frac{s}{s-u}\, ds$ ⇝ we need to analytically continue to $|u| \leq 1$ (only resonances at $|u| \leq 1$)

$\displaystyle\int_{-1}^{1} 1 + \frac{u}{s-u}\, ds = 2 + u\int_{-1}^{1}\frac{ds}{s-u}$



$\displaystyle\int_{-1}^{1}\frac{ds}{s-u} = PV\int \frac{ds}{s-u} + \mathrm{Res}\{s=u\}$

$\displaystyle = \ln(s-u)\Big|_{-1}^{1} + \pi i(1)$

$\displaystyle \Rightarrow \epsilon(\omega,k) = 1 + \frac{1}{2k^2\lambda_D^2}\left[2 + u\ln\left(\frac{1-u}{-1-u}\right) + i\pi u\right]$

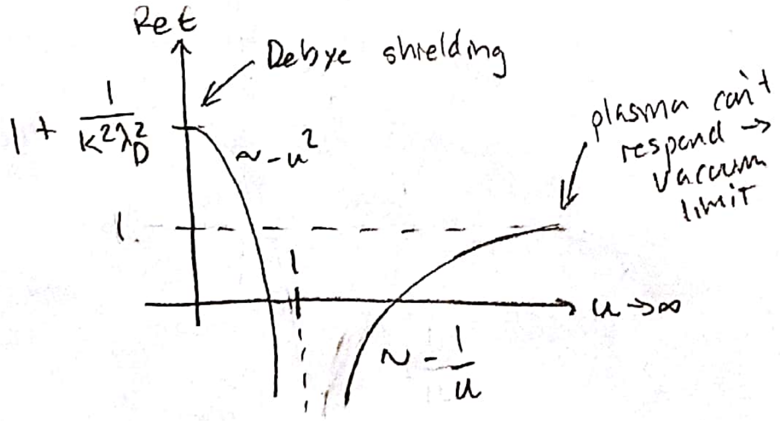
so
$$\boxed{\begin{aligned} \mathrm{Re}\,\epsilon &= 1 + \frac{1}{k^2\lambda_D^2}\left[1 + \frac{u}{2}\ln\left(\frac{u-1}{u+1}\right)\right] \\ \mathrm{Im}\,\epsilon &= \frac{\pi}{2}\frac{u}{k^2\lambda_D^2} \quad (|u| \leq 1) \end{aligned}}$$

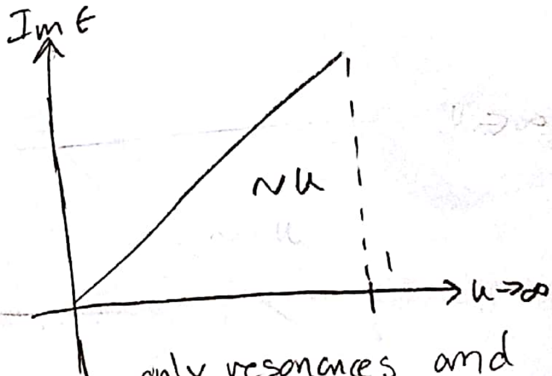
At $u=1$, $\mathrm{Re}\,\epsilon \sim \ln(0)$ → logarithmic singularity.

(c.) $u \ll 1$, $\ln(1-u) \sim 0 - u - \frac{u^2}{2}$, $\ln(1+u) \sim 0 + u - \frac{u^2}{2}$

$u(\ln(1-u) - \ln(1+u)) \sim -2u^2$ $\qquad \ln(1\frac{1}{t}) \sim t$

$u \gg 1$, $\frac{1}{t}(\ln(1-\frac{1}{t}) - \ln(1+\frac{1}{t})) \sim u\left[0 - \frac{2}{u} - \frac{u}{u^2}\right) \sim -2\left(1 + \frac{2}{u}\right)$





only resonances and Landau damping for $u < 1$