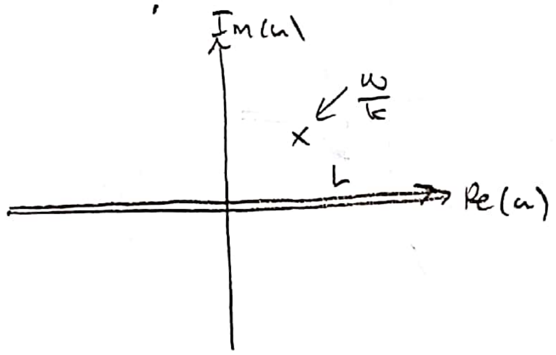
2012 I : Q4 Waves

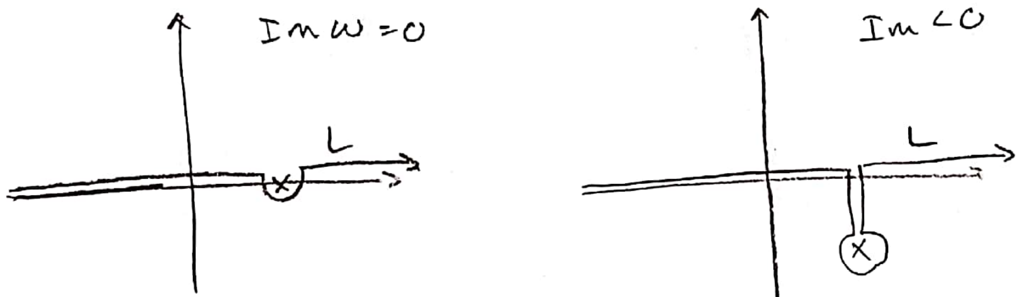
$$1 + \frac{\omega_{pe}^2}{k^2}\int_L \frac{1}{\omega/k - u}\frac{\partial g_{eo}}{\partial u}\,du = 0 \quad \text{where } g_{eo} \equiv \frac{1}{n_{eo}}\int dv_y\,dv_z\,f_{eo}$$

a.) The Landau contour lets you deal with the poles in $\mathcal{D}(\omega,k)$.

At Im > 0, the Landau contour follows the real axis:



To analytically continue to Im ≤ 0, you have to bend the contour:



b.) Assume $Im(\omega) > 0$.

$$1 + \frac{\omega_{pe}^2}{k^2}\int_L \frac{1}{\frac{\omega}{k} - u}\frac{\partial g_0}{\partial u}\,du = 1 + \frac{\omega_{pe}^2}{k^2}\int_L du\,\frac{k}{\omega}\,\frac{\left(\frac{\omega}{k}-u\right)+u}{\frac{\omega}{k}-u}\,g'(u)$$

$$= 1 + \frac{\omega_{pe}^2}{k^2}\frac{k}{\omega}\int_L du\left[\cancelto{0}{g'(u)} + \frac{ug'}{\frac{\omega}{k}-u}\right]\left(\frac{\frac{\omega^*}{k}-u}{\frac{\omega^*}{k}-u}\right)$$

$$= 1 + \frac{\omega_{pe}^2}{k^2}\int_L \frac{\frac{k}{\omega}\left(\frac{\omega^*}{k}-u\right)}{\left|\frac{\omega}{k}-u\right|^2}\,ug'\,du$$

$$\Rightarrow\ 0 = \underset{Im(\omega)}{\omega} + \underset{-Im(\omega)}{\omega^*}\,\frac{\omega_{pe}^2}{k^2}\Big(\int_L \frac{ug'}{\left|\frac{\omega}{k}-u\right|^2}\,du - \int_L \frac{ku^2g'}{\left|\frac{\omega}{k}-u\right|^2}\,du\Big)$$

keeping only the imaginary part yields:

$$0 = \mathrm{Im}(\omega)\left[1 - \frac{\omega_p^2}{k^2}\int_L \frac{ug'}{|\frac{\omega}{k} - u|^2}\,du\right] - 0$$

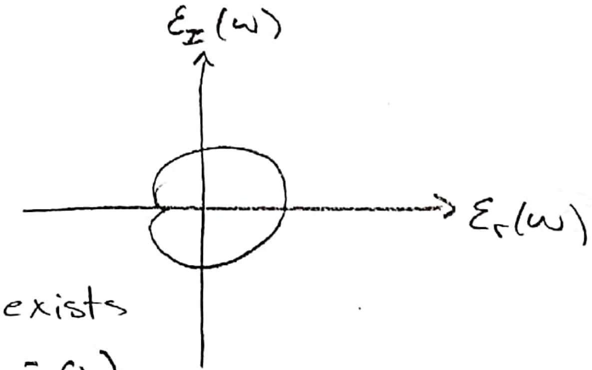
By assumption $ug' < 0$. $|\frac{\omega}{k} - u|^2 > 0$. $\frac{\omega_p^2}{k^2} > 0$.

which gives $\mathrm{Im}(\omega)[1 + c] = 0$, $c > 0$,

which is only satisfied if $\mathrm{Im}(\omega) = 0$, which corresponds to a stable system.

Better solution:

Use Nyquist theorem:



Instability if there exists a point $(\varepsilon_r < 0, \varepsilon_I = 0)$

$$\varepsilon = 1 + \frac{\omega_{pe}^2}{k^2}\int \frac{1}{\omega/k - u}\frac{\partial g}{\partial u}\,du$$

$$\varepsilon_r = 1 + \frac{\omega_p^2}{k^2}\,\mathrm{pv}\int (\quad) \qquad \varepsilon_I = \frac{\omega_p^2}{k^2}\left(\frac{\partial g}{\partial u}\bigg|_{\omega/k}\right)\frac{2\pi}{2}$$

$\varepsilon_I = 0 \Rightarrow \frac{\partial g}{\partial u}\Big|_{\omega/k} = 0$ which is true at $u = 0$, $u = \pm\infty$

(since $\frac{\partial}{\partial u} g(0) = 0$)

$$\varepsilon_R = 1 + \frac{\omega_{pe}^2}{k^2}\int \frac{\frac{\partial}{\partial u}[g(u) - g(0)]}{\omega/k - u}\,du$$

$$\text{IBP} \Rightarrow 1 + \frac{\omega_{pe}^2}{k^2}\int \frac{-1}{(\omega/k - u)^2}[g(u) - g(0)]\,du$$

$$\varepsilon_R = 1 + \underbrace{\frac{\omega_p^2}{k^2}}_{>0}\int \frac{\overbrace{[g(0) - g(u)]}^{>0 \text{ since } g \text{ monotonically decreasing}}}{\underbrace{(\omega/k - u)^2}_{>0}}\,du$$

$\Rightarrow \varepsilon_R > 0$ so stable.