2013 II : Q2 Waves

a.) $\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \frac{\vec{E}}{m} \cdot \nabla_v f = 0$ let $f = n_0 f_0 + f_1$

$\int d^3v \; f_0(\vec{v}) = 1$

$$-i\omega f_1 + v \cdot ik f_1 + n_0 \frac{q}{m}\left[\tilde{E} + \frac{v \times \tilde{B}}{c}\right] \cdot \frac{\partial f_0}{\partial v} = 0 \qquad \frac{-n_0}{ik - \omega}$$

$$\Rightarrow f_1 = n_0 \frac{q}{m}\left[\tilde{E} + \frac{v \times \tilde{B}}{c}\right]\left(\frac{-i}{\omega - k \cdot v}\right) \cdot \frac{\partial f_0}{\partial v}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \Rightarrow \cancel{i}\vec{k} \times \vec{E} = +\frac{1}{c}\cancel{i}\omega \vec{B} \Rightarrow \tilde{B} = \frac{c}{\omega}(k \times \tilde{E})$$

$\vec{A} \times (\nabla \times \vec{B}) = (\nabla \vec{B}) \cdot \vec{A} - (\vec{A} \cdot \nabla)\vec{B}$

$$\Rightarrow f_1 = n_0 \frac{q}{m} \frac{-i}{\omega - k \cdot v}\left[\tilde{E} + \frac{1}{\omega} v \times k \times \tilde{E}\right] \cdot \frac{\partial f_0}{\partial v}$$

$$= \frac{n_0 q}{m} \frac{-i}{\omega - k \cdot v}\left[E_j + \frac{1}{\omega}\left(k_\ell E_\ell v_j - v_\ell k_\ell E_j\right)\right] \cdot \frac{\partial f_0}{\partial v}$$

$$= \frac{n_0 q}{m} \frac{-i}{\omega - k \cdot v}\left[E_j\left(1 - \frac{v_\ell k_\ell}{\omega}\right) + \frac{k_\ell v_j}{\omega} E_\ell\right] \cdot \frac{\partial f_0}{\partial v}$$

$$= \frac{n_0 q}{m} \frac{-i}{\omega - k \cdot v}\left[\delta_{\ell j}\left(1 - \frac{vk}{\omega}\right) + \frac{k_\ell v_j}{\omega}\right] E_\ell \frac{\partial f_0}{\partial v_\ell}$$

$$\vec{J}_\ell = \int q v_\ell f_1 \, d^3v = \underbrace{\frac{n_0 q^2}{m} \int \frac{-i v_\ell}{\omega - kv}\left[\delta_{\ell j}\left(1 - \frac{vk}{\omega}\right) + \frac{k_\ell v_j}{\omega}\right] \frac{\partial f_0}{\partial v_\ell}}_{\sigma} E_\ell$$

$$\varepsilon_{ij} = \delta_{ij} + \frac{4\pi i \sigma_{ij}}{\omega} = \delta_{ij} + \frac{\omega_p^2}{\omega^2} \int \frac{v_\ell}{1 - kv/\omega}\left[\underbrace{\delta_{\ell j}\left(1 - \frac{vk}{\omega}\right)}_{①} + \underbrace{\frac{k_\ell v_j}{\omega}}_{②}\right] \frac{\partial f_0}{\partial v_\ell} d^3v$$

$$\int \frac{v_\ell}{\cancel{1 - kv/\omega}} \delta_{\ell j} \cancel{\left(1 - \frac{vk}{\omega}\right)} \frac{\partial f_0}{\partial v_\ell} d^3v = \delta_{ij} \int v_i \frac{\partial f_0}{\partial v_i} d^3v$$

$$= \delta_{ij}\left[\cancel{v_i f_0}\Big|_{-\infty}^{\infty} \to 0 - \int f_0 \, d^3v\right] = -\delta_{ij}$$

$$\Rightarrow \varepsilon_{ij} = \delta_{ij} + \frac{\omega_p^2}{\omega^2}\left[-\delta_{ij} + \int d^3v \, \frac{k_i v_i v_j}{\omega - k v_z} \frac{\partial f_0}{\partial v_z}\right]$$

$$\Rightarrow \varepsilon_{ij}(\omega,\vec{k}) = \delta_{ij}\left(1-\frac{\omega_p^2}{\omega^2}\right) + \frac{\omega_p^2}{\omega^2}\int \frac{k v_i v_j}{\omega - k v_z}\frac{\partial f_0}{\partial v_z}\, d^3v$$

(b.) Assume $f_0(\vec{v}) = \frac{1}{\pi^{3/2} w_\perp^2 w_{\parallel}} \exp\left(-\frac{v_x^2}{w_\perp^2} - \frac{v_y^2}{w_\perp^2} - \frac{v_z^2}{w_\parallel^2}\right)$ $\quad w_\perp^2 = \frac{2T_\perp}{m},\ w_\parallel^2 = \frac{2T_\parallel}{m}$

$$\int \frac{k v_i v_j}{\omega - k v_z}\frac{\partial f_0}{\partial v_z}\, d^3v = \int \frac{k v_i v_j}{\omega - k v_z}\left(-\frac{2v_z}{w_\parallel^2}\right) f_0(\vec{v})\, dv_x\, dv_y\, dv_z$$

off diagonal terms have $v_i \neq v_j$ so then the integral can be written as $\int_{-\infty}^{\infty} f_{odd} f_{even}\, dv = 0$. The $v_x v_x v_z$, v_z^3 and $v_y^2 v_z$ terms surive beause of the pole $\frac{1}{\omega - k v_z}$. $\therefore \varepsilon_{ij}$ is diagonal.

$$\varepsilon_\perp = \varepsilon_{xx} = \varepsilon_{yy} = 1 - \frac{\omega_p^2}{\omega^2} - \frac{2\omega_p^2}{w_\parallel^2 \omega^2}\iiint \frac{k v_x^2 v_z}{\omega - kz} f_0(\vec{v})\, dv_x\, dv_y\, dv_z$$

$$\int dv_x \to \frac{\sqrt{\pi}}{2} w_\perp^3 \qquad \int dv_y \to \sqrt{\pi}\, w_\perp$$

$$\Rightarrow \varepsilon_\perp = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\cancel{2}\,\omega_p^2}{w_\parallel^2\,\omega^2}\frac{\cancel{\pi}}{\cancel{2}} w_\perp^2 \frac{\cancel{w_\perp^2}}{\cancel{\pi^{3/2}}\,\pi^{1/2}\cancel{w_\perp^2}\, w_\parallel}\int_{-\infty}^{\infty} \frac{k v_z}{\omega - k v_z}\exp\left(-\frac{v_z^2}{w_\parallel^2}\right) dv_z$$

let $t = \frac{v_z}{w_\parallel}$ then $dv_z = w_\parallel dt$, $v_z = w_\parallel t$

$$\int_{-\infty}^{\infty} \frac{k w_\parallel^2 t}{\omega - k w_\parallel t} e^{-t^2} dt = \int_{-\infty}^{\infty} \frac{1}{\cancel{k w_\parallel}} \frac{\cancel{k} w_\parallel^2 t}{\frac{\omega}{k w_\parallel} - t} e^{-t^2} dt = w_\parallel \int -\frac{t e^{-t^2}}{t - \xi} dt$$

$$\xi = \frac{\omega}{k w_\parallel}$$

IBP w/ $\quad u = \frac{-1}{t - \xi} \qquad v = -\frac{1}{2}e^{-t^2}$

$\qquad du = \frac{1}{(t-\xi)^2} \qquad dv = t e^{-t^2} dt$

$$\Rightarrow w_\parallel \left[\cancel{\frac{e^{-t^2}}{2(\xi - t)}}\Bigg|_{-\infty}^{\infty \to 0} + \frac{\sqrt{\pi}}{2}\frac{dZ}{d\xi} \right]$$

$$\Rightarrow \varepsilon_\perp = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_p^2}{\omega^2}\left(\frac{w_\perp^2}{w_{\parallel}^2}\right)\left[\frac{1}{2}\frac{\partial Z}{\partial \varsigma}\right]$$

$$= 1 - \frac{\omega_p^2}{\omega^2} + \frac{T_\perp}{T_\parallel}\frac{\omega_p^2}{\omega^2}\left[1 + \varsigma Z(\varsigma)\right]$$

Dispersion relation: $N_i N_j - N^2 \delta_{ij} + \varepsilon_{ij} = 0$

note $k_x = k_y = 0$ so for $D_{xx} = D_{yy} = D_\perp$ we have

$$-N^2 + \varepsilon_{ij} = 0 \Rightarrow \boxed{\omega^2 - c^2k^2 - \omega_p^2 + \omega_p^2\left(\frac{T_\perp}{T_\parallel}\right)\left[1 + \varsigma Z(\varsigma)\right] = 0}$$

(c.) For hot plasma, $\varsigma \ll 1 \Rightarrow Z(\varsigma) \sim i\pi^{1/2}\exp(-\varsigma^2) - 2\varsigma + \frac{4}{3}\varsigma^3 + \ldots$

$$\Rightarrow \omega^2 = c^2k^2 + \omega_p^2\left[1 - \frac{T_\perp}{T_\parallel}\left(1 + i\pi^{1/2}\varsigma e^{-\varsigma^2} - 2\varsigma^2\right)\right]$$

take $e^{-\varsigma^2} \to 1$, $\varsigma^2 = \frac{\omega^2}{k^2 w_\parallel^2} \to 0$, $\frac{\omega^2}{c^2k^2} \to 0$

$$0 = c^2k^2 + \omega_p^2\left[1 - \frac{T_\perp}{T_\parallel}\left(1 + i\pi^{1/2}\frac{\omega}{k w_\parallel}\right)\right]$$

$$\left(1 + i\pi^{1/2}\frac{\omega}{k w_\parallel}\right) = \frac{T_\parallel}{T_\perp \omega_p^2}\left(c^2k^2 + \omega_p^2\right)$$

$$\omega = i\frac{k w_\parallel}{\pi^{1/2}}\left[1 - \frac{T_\parallel}{T_\perp}\left(\frac{c^2k^2}{\omega_p^2} + 1\right)\right]$$

If $T_\perp > T_\parallel$, $\mathrm{Im}(\omega) > 1$ for small enough k → unstable!

(d.) For cold plasma, $\varsigma \gg 1 \Rightarrow Z(\varsigma) \sim i\pi^{1/2}\exp(-\varsigma^2) - \frac{1}{\varsigma} - \frac{1}{2\varsigma^3} + \ldots$

$$\Rightarrow \omega^2 = c^2k^2 + \omega_p^2 - \omega_p^2\frac{T_\perp}{T_\parallel}\left(1 - 1 - \frac{1}{2}\frac{1}{\varsigma^2}\right)$$

$$\omega^2 = c^2k^2 + \omega_p^2 + \omega_p^2\frac{T_\perp}{T_\parallel}\frac{k^2 w_\parallel^2}{2\omega^2} \qquad \frac{w_\parallel^2}{2T_\parallel} = \frac{1}{m}$$

$$\Rightarrow \omega^4 - \omega^2(c^2k^2+\omega_p^2) - \omega_p^2 T_\perp \frac{k^2}{m}$$

$$\omega^2 = \frac{1}{2}\left(c^2k^2+\omega_p^2 \pm \sqrt{(c^2k^2+\omega_p^2)^2 + 4T_\perp \frac{k^2}{m}\omega_p^2}\right)$$

$$\Rightarrow \omega = \pm\sqrt{\frac{1}{2}\left(c^2k^2+\omega_p^2 \pm \sqrt{(c^2k^2+\omega_p^2)^2 + 4T_\perp\omega_p^2\frac{k^2}{m}}\right)}$$

NOTE: $(c^2k^2+\omega_p^2) < \left[(c^2k^2+\omega_p^2)^2 + 4T_\perp\omega_p^2\frac{k^2}{m}\right]^{1/2}$

so there exists one mode with $\mathrm{Im}(\omega) > 0$.

This mode is unstable.

The cold plasma approximation is $\frac{\omega}{k w_{\parallel}} >> 1$

$$\omega^2 = \frac{1}{2}(c^2k^2+\omega_p^2)\left(1 \pm \sqrt{1 - \frac{2w_\perp^2k^2\omega_p^2}{(\omega_p^2+c^2k^2)^2}}\right)$$

$$\approx \frac{1}{2}(c^2k^2+\omega_p^2)\left(1 \pm \left(1 \mp \frac{2w_\perp^2k^2\omega_p^2}{(\omega_p^2+c^2k^2)^2}\right)\right)$$

$$\omega^2 \sim \frac{w_\perp^2k^2\omega_p^2}{\omega_p^2+c^2k^2}$$

$$\Rightarrow \xi^2 \sim \frac{1}{\cancel{k^2}w_\parallel^2}\frac{w_\perp^2\cancel{k^2}\omega_p^2}{\omega_p^2+c^2k^2} \sim \frac{w_\perp^2}{w_\parallel^2} \sim \frac{T_\perp}{T_\parallel} >> 1$$

$$\Rightarrow \boxed{T_\perp >> T_\parallel}$$

Wowee this was a long one...