2013 II : Q3 MHD

1.) $\delta(W - \mu K) = 0 = \delta W - \mu \delta K$

$$\frac{1}{2}\int 2V\delta V + 2B\delta B \, d\tau - \mu \int V\delta B + B\delta V \, d\tau = 0$$

$$\int d\tau \, (V - \mu B)\delta V + (B - \mu V)\delta B$$

$$\Rightarrow V - \mu B = 0 \qquad B - \mu V = 0$$

$$V - \mu^2 V = 0 \quad \Rightarrow \boxed{\mu = \pm 1}, \ \boxed{V = \pm B}$$

2.) $\delta(W - \lambda H) = 0$

$$\delta H = \frac{1}{2}\int d\tau \ \delta A \cdot (\nabla \times A) + A \cdot (\nabla \times \delta A)$$

$$\Rightarrow \int d\tau \ V\delta V + B\delta B - \frac{\lambda}{2}\left[2A(\underbrace{\nabla \times \delta A}_{\delta B}) - \cancel{\nabla \cdot (\delta A \times A)}\right) = 0$$

goes to zero by divergence thrm.

$$\int d\tau \ V\delta V + (B - \lambda A)\delta B = 0$$

$$\Rightarrow \boxed{V = 0}, \ B = \lambda A \Rightarrow \boxed{\nabla \times B = \lambda B}$$

This is a force - free equilibria! ($J \times B = \nabla \times B \times B = B \times B = 0$)

3.) $\delta(W - \mu K - \lambda H) = 0$

$$\Rightarrow \int d\tau \ (V - \mu B)\delta V + (B - \mu V - \lambda A)\delta B = 0$$

$$\Rightarrow \boxed{V = \mu B}, \ B = \mu V + \lambda A \Rightarrow B = \mu^2 B + \lambda \nabla \times B$$

$$\boxed{B = \frac{\lambda}{1 - \mu^2} \nabla \times B}$$

Again, this is a force - free equilibria!

4.) $B = \frac{\lambda}{1-\mu^2} \nabla \times B$

as $\mu \to 0 \Rightarrow B = \lambda \nabla \times B$, $V = 0$ ✓

as $\lambda \to 0 \Rightarrow B(1-\mu^2) = 0 \Rightarrow V = \mu B$ ✓

with $\mu \to 0$ and $\lambda \to 0 \Rightarrow V = 0$, $B = 0$.

$\Rightarrow$ unconstrained minimization will just kill off both of the fields.