2013 I: SA Math

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+xy=0 \quad \Rightarrow \quad xy''+2y'+xy=0$$

a.) $x\to 0$: regular singular point $\Rightarrow$ use $y=\sum_n x^r a_n x^n$

$$\sum_n a_n(n+r)(n+r-1)x^{n+r-1}+2a_n(n+r)x^{n+r-1}+a_n x^{r+n+1}=0$$

$$\sum_{n=0}^{\infty} a_n(n+r)\left[(n+r-1)+2\right]x^{n+r-1}+\sum_{n=2}^{\infty} a_{n-2}x^{n+r-1}=0$$

$(n=0)$: $a_0(r)(r+1)=0 \Rightarrow r=0,\ r=-1$

For $r=0$, $(n\geq 2) \Rightarrow a_n n(n+1)+a_{n-2}=0$

$$a_n=\frac{a_{n-2}}{n(n+1)},\quad a_0=1,\quad a_1=0$$

$$\Rightarrow y_1(x)=\sum_{n=0}^{\infty}\frac{x^n}{(n+1)!},\ n \text{ even} \quad \Rightarrow \quad \boxed{y_1(x)=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n+1)!}}$$

$$a_n(r)=\frac{a_{n-2}(r)}{(n+r)(n+r+1)}$$

Since $r_1-r_2=s>0$, $s\in\mathbb{Z}$

$$y_2(x)=y_1(x)\ln(x)+x^{r_2}\sum_{n=0}^{\infty}\left.\frac{\partial(r-r_2)a_n(r)}{\partial r}\right|_{r=s}x^n$$

$$\Rightarrow \boxed{y_2(x)=\ln(x)\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n+1)!}+\frac{1}{x}\sum_{n=0}^{\infty}b_n x^n}$$

where $b_n=\left[\frac{\partial}{\partial r}\frac{(r+1)b_{n-2}(r)}{(n+r)(n+r+1)}\right]_{r=-1}$

b.) $x\to\infty$ $\quad t=\frac{1}{x}$ $\quad \frac{dy}{dx}=\frac{dt}{dx}\frac{dy}{dt}=-\frac{1}{x^2}\dot{y}=-t^2\dot{y}$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{dt}{dx}\right)\frac{dy}{dt}+\frac{dt}{dx}\frac{d}{dx}\left(\frac{dy}{dt}\right)=\frac{2}{x^3}\dot{y}+\frac{1}{x^4}\ddot{y}$$

$$\frac{dt}{dx}\frac{d}{dt}\frac{dy}{dt}$$

$$=2t^3\dot{y}+t^4\ddot{y}$$

$$\Rightarrow \frac{1}{t}\left(2t^3 \dot{y} + t^4 \ddot{y}\right) + 2t\left(-t^2 \dot{y}\right) + \frac{1}{t} y = 0$$

$$\Rightarrow t^3 \ddot{y} + \left(2t^2 - 4t^3\right)\dot{y} + \frac{1}{t} y = 0$$ $\Rightarrow$ irregular singular point

try $y(x) \sim e^{S(x)}$

$$x S'' + x (S')^2 + 2S' + x = 0$$

$x(S')^2 + x = 0 \Rightarrow S' = \pm i$ satisfies assumptions $\Rightarrow S(x) = \pm i x$

let $S'(x) = \pm i + g'(x)$ to get

$$x(g'') + x\left(\cancel{-1} \pm 2ig' + \underbrace{\cancel{(g')^2}}_{g' \ll 2i}\right) \pm 2i + \underbrace{\cancel{2g'}}_{g' \ll 2i} \cancel{+x} = 0$$

$$\pm 2ixg' \pm 2i = 0 \Rightarrow g' = -\frac{1}{x} \Rightarrow g(x) = \ln\left(-\frac{1}{x}\right)$$

$$\Rightarrow \boxed{y_1(x) \sim -\frac{1}{x} e^{ix}, \quad y_2(x) \sim -\frac{1}{x} e^{-ix}}$$

(c.) Try $y(x) = \int_C f(t) e^{xt}\, dt$, then

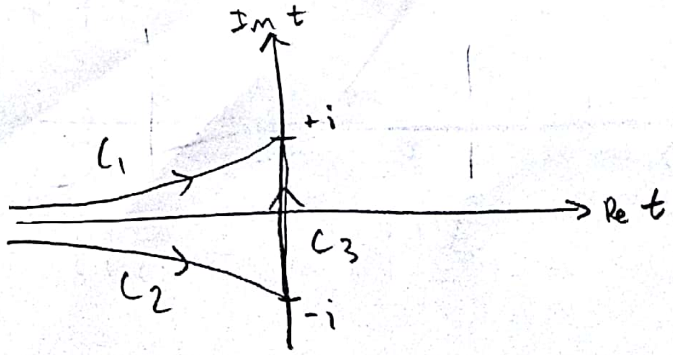
$$x\int_C t^2 f(t) e^{xt}\, dx + 2\int_C t f(t) e^{xt} + x\int_C f(t) e^{xt} = 0$$

$$\int_C dt\left[-2tf - t^2 f' + 2tf - f'\right]e^{xt} + \left[\left(t^2 f(t) + f(t)\right)e^{xt}\right]_C = 0$$

$\Rightarrow (t^2+1) f' = 0 \Rightarrow f(t) = A$ where $A = $ const.

$$\Rightarrow \left[A(t^2+1)e^{xt}\right]_C = 0$$

vanish @ $t = \pm i, -\infty$



Any of C_1, C_2, C_3 will work. NOTE: $C_1 + C_2 = C_3$

so $\boxed{y_1(x) = \int_{-\infty}^{i} e^{xt}\, dt, \quad y_2(x) = \int_{-\infty}^{-i} e^{xt}\, dt}$

(d.) If $y(0) = 1$, we have $y(0) = \int_C dt$.

Only contour C_3 has finite length: $\int_{-i}^{i} dt = 2$

so then $\boxed{y(x) = \frac{1}{2} \int_{-i}^{i} e^{xt}\, dt}$