2014 II: QS Waves

(a.) $\mathcal{E} = 1 + \sum_s \chi_s$ "electron plasma" so $s = e$

non-Maxwellian

$$\Rightarrow \mathcal{E} = 1 - \frac{\omega_{pe}^2}{k^2}\int \frac{f_0'}{v-\omega/k}\,dv - \frac{\omega_{pe}^2}{k^2}\int \frac{F_0'}{v-\omega/k}\,dv$$

let F_0' term $\to 0$ in $\mathcal{E}_R$ because "low-density beam"

$$\Rightarrow \mathcal{E}_R = 1 - \frac{\omega_{pe}^2}{k^2} PV \int \frac{f_0'}{v-\omega/k}\,dv \qquad \frac{\omega}{k} \gg v \qquad \frac{vk}{\omega} \ll 1$$

$$= 1 - \frac{\omega_{pe}^2}{k^2}\int dv\, f_0' \, \frac{k}{\omega}\frac{1}{\left(\frac{vk}{\omega}-1\right)} \qquad \frac{1}{x-1} \approx -1 + x\left(-\cancel{1}^{\,-1}\right)$$

$$\approx 1 - \frac{\omega_{pe}^2}{k^2}\int dv\, f_0' \, \frac{k}{\omega}\left(-1 - \frac{vk}{\omega}\right) \qquad \frac{d}{dv}\left(1 + \frac{vk}{\omega}\right) = \frac{k}{\omega}$$

IBP $\Rightarrow$

$$= 1 + \frac{\omega_{pe}^2}{k^2}\frac{k}{\omega}\left[\ \cancel{\Big|_{-\infty}^{\infty}}^{\,0} - \int dv\, \frac{k}{\omega} f_0(v) \right. \qquad dv \int f_0(v) = 1$$

$$\Rightarrow \mathcal{E}_R = 1 - \frac{\omega_p^2}{\omega^2} \approx 0 \Rightarrow \boxed{\omega_R^2 = \omega_{pe}^2}$$

Similarily f_0' term $\to 0$ in $\mathcal{E}_I$ because instability comes from beam.

$\omega_R \gg \omega_I$

$$\mathcal{E}_I = -\frac{\omega_{pe}^2}{k^2}(\pi i)\,\mathrm{Res}\left\{F_0'(v) \text{ @ } v = \omega/k\right\} = -\frac{\omega_{pe}^2}{k^2}(i\pi) F_0'\left(\omega/k\right)$$

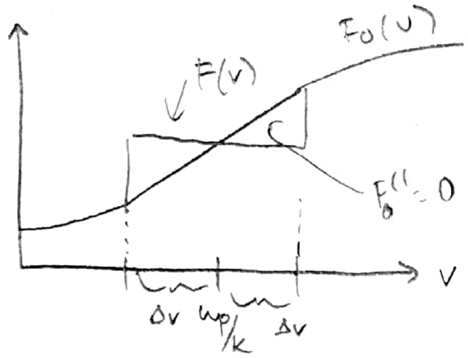
Then note $\mathcal{E}(\omega) = \mathcal{E}_R(\omega) + i\mathcal{E}_I(\omega)$

$\ll \mathcal{E}_R(\omega_R)$

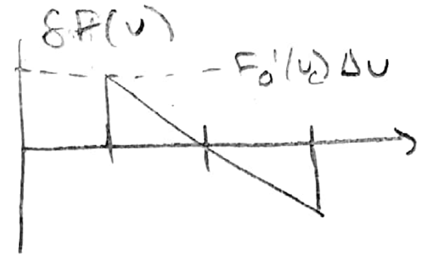
$$0 \approx \cancel{\mathcal{E}_R(\omega_R)}^{\,0} + i\mathcal{E}_I(\omega_R) + i\omega_I\, \partial_\omega\left(\mathcal{E}_R(\omega_R) + i\cancel{\mathcal{E}_I}(\omega_R)\right)$$

$$\Rightarrow \omega_I = -\frac{\mathcal{E}_I(\omega_R)}{\partial_\omega(\mathcal{E}_R(\omega_R))} \Rightarrow \boxed{\omega_I = \pi \frac{\omega_{pe}^2}{k^2} F_0'\left(\frac{\omega_R}{k}\right)}$$

(b.)

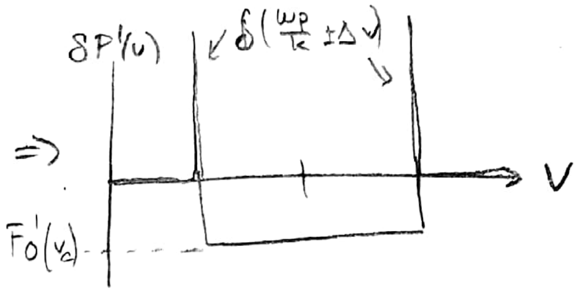


$\Rightarrow$



$$V_c = \frac{\omega p}{k}$$

$\Rightarrow$



$$\varepsilon = \bar{\varepsilon} + \delta\epsilon$$

$$\delta\varepsilon = 1 - \frac{\omega_{pe}^2}{k^2}\int dv \, \frac{\delta F'(v)}{v-\omega/k}$$

$\Rightarrow A = \text{normalization} = F_0'(v)\Delta v$

$$\delta F'(v) = A\left[\delta\left(v - \frac{\omega p}{k} + \Delta v\right) + \delta\left(v - \frac{\omega p}{k} - \Delta v\right)\right] - F_0'(v_c)$$

$$= \left[\Delta v\left[\delta(+) + \delta(-)\right] - 1\right] F_0'(v_c)$$

$$\delta\varepsilon = 1 - \frac{\omega_{pe}^2}{k^2}\int dv \, \frac{\Delta v F_0'(v_c)}{v-\omega/k}\left[\delta(+) + \delta(-)\right] + \frac{\omega_{pe}^2}{k^2}\int_{\frac{\omega_{pe}}{k}-\Delta v}^{\frac{\omega_{pe}}{k}+\Delta v} dv \, \frac{F_0'(v_c)}{v-\omega/k}$$

$$= 1 - \frac{\omega_{pe}^2}{k^2}\Delta v\left[\frac{F_0'(\frac{\omega p}{k})}{\underbrace{\frac{\omega p}{k}}_{v_c} - \frac{\omega}{k} + \Delta v} + \frac{F_0'(\frac{\omega p}{k})}{\underbrace{\frac{\omega p}{k}}_{V_c} - \frac{\omega}{k} - \Delta v}\right] + \frac{\omega_{pe}^2}{k^2} F_0'(V_c) \ln\left(v - \frac{\omega}{k}\right)\Big|_{v_c-\Delta v}^{v_c+\Delta v}$$

$$= 1 - \frac{\omega_{pe}^2 F_0'(v_c)}{k^2}\left[\frac{\Delta v^2}{k\,\Delta v}\left[\frac{\overbrace{(kv_c - \omega - \cancel{k\Delta v}) + (kv_c - \omega + \cancel{k\Delta v})}^{2\omega}}{(kv_c - \omega - k\Delta v)(kv_c - \omega + k\Delta v)}\right] - \ln\left(\frac{V_c + \Delta v - \frac{\omega}{k}}{V_c - \Delta v - \frac{\omega}{k}}\right)\right]$$

$$w^2 - 1 = \frac{\omega^2 - 2kv_c\omega + k^2v_c^2 - k^2\Delta v^2}{k^2\Delta v^2} \quad // \quad \text{if } \omega k\Delta v \ll k^2\Delta v^2, \; k^2 v_c \Delta v \ll k^2\Delta v^2$$

$$\frac{kv_c + k\Delta v - \omega}{kv_c - k\Delta v - \omega} \Rightarrow \frac{w+1}{w-1}$$

$$= 1 - \frac{\omega_{pe}^2 F_0'(v_c)}{k^2}\left[\ln(1-w) - \ln(1+w) + \frac{2w}{w^2-1}\right]$$

$$\varepsilon = \bar{\varepsilon} + \delta\varepsilon. \quad \bar{\varepsilon} = 1 - \frac{\omega p^2}{\omega^2} = 1 - \frac{\omega_{pe}^2 F_0'(v_c)}{k^2}\left(\frac{k^2}{F_0'(v_c)\omega^2}\right)$$

$$= 1 - \frac{\omega_{pe}^2 F_0'(v_c)}{k^2}\left[\frac{k^2}{F_0'(v_c)\,(k\Delta v\, w + k\underbrace{v_c}_{\Delta v})}\right]$$

$$\Rightarrow \boxed{\varepsilon = 1 - \frac{\omega_p^2}{k^2 v_c^2}, \quad \beta = \frac{\omega_p^2}{k^2} F_0'(v_c), \quad \nu = \frac{2\Delta v}{v_c^3 F_0'}}$$

$$\Rightarrow \boxed{\varepsilon + \beta\left(\nu w + \ln(1-w) - \ln(1+w) + \frac{2w}{w^2-1}\right) = 0}$$

(c.) Let $\varepsilon = 0$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$

1.) Let $\omega \ll 1$

$$\Rightarrow \nu w - w - \cancel{\frac{w^2}{2}} - \frac{w^3}{3} - w + \cancel{\frac{w^2}{2}} - \frac{w^3}{3} + 2w - 2w^3 = 0$$

$$(\nu - 4)w - \frac{8}{3}w^3 = 0 \qquad -\frac{2}{3} - \frac{6}{3} = -8/3$$

$$\Rightarrow \omega^2 = \frac{3}{8}(\nu - 4)$$

$\Rightarrow$ instability for $\nu < 4$

Not suppressed even though $F'(\omega_r/k) = 0$ because ν depends only on $F_0'(\omega_R/k)$.

2.) Let $\nu \ll 1$ $\Rightarrow \ln\left(\frac{1-\omega}{1+\omega}\right) + \cancel{\frac{2\omega}{\omega^2-1}} \to 0$ as $\omega \to \pm i\infty$ $= -\nu w$

$$\Rightarrow \ln(-w) - \ln(\omega) = -\nu w$$

$$\nu w + \cancel{\ln|w|} + i\arg(-w) - \cancel{\ln|w|} + i\arg(w) = 0$$

If $\omega_i > 0 \Rightarrow \nu w = +i\pi/2 + i\pi/2$

If $\omega_i < 0 \Rightarrow \nu w = -i\pi/2 - i\pi/2$

$\Rightarrow W_I = \pm\frac{\pi}{\nu}$

$$\boxed{W_\pm = \pm\frac{\pi\, v_c^3 F_0'}{2\Delta v}}$$