2014 I : Q2 Waves

$\vec{k} \parallel \vec{B}_0$, $n \gg 1$, $\omega \ll |\Omega_e|$, $\Omega_e = \Omega_e(z)$, $\omega_p = \omega_p(z)$

a.) The geometrical optics equations are

$$\begin{cases} d_t \vec{x} = \partial_k \omega \\ d_t \vec{k} = -\partial_x \omega \\ d_t \omega = \partial_t \omega \end{cases} \qquad D(t, \vec{x}, \omega, \vec{k}) = 0 \qquad \partial_t \vec{I} + \nabla(v_g \vec{I}) = 2\gamma \vec{I}$$

$$\vec{I} = \frac{1}{16\pi\omega}\left[\tilde{E}_c^{*} \partial_\omega (\omega \epsilon_H) \tilde{E}_c + |\tilde{B}_c|^2\right]$$

For Whistler waves, $N^2 = R = 1 - \frac{\omega_{pe}^2}{\omega(\omega + \Omega_e)} = 1 - \frac{\omega_{pe}^2}{\omega(\omega - |\Omega_e|)} \approx 1 + \frac{\omega_{pe}^2}{\omega|\Omega_e|}$

$$\Rightarrow \frac{c^2k^2}{\omega^2} \approx 1 + \frac{\omega_{pe}^2}{\omega|\Omega_e|} \overset{n\gg1}{\Rightarrow} \frac{c^2k^2}{\omega^2} \approx \frac{\omega_{pe}^2}{\omega|\Omega_e|} \Rightarrow \omega \approx c^2k^2 \frac{|\Omega_e|}{\omega_{pe}^2}$$

$$\Rightarrow \boxed{\begin{cases} \frac{dz}{dt} = 2c^2k\frac{|\Omega_e|}{\omega_{pe}^2} = v_g \\ \frac{dk}{dt} = -c^2k^2\left(\frac{\partial}{\partial z}\frac{|\Omega_e|}{\omega_{pe}^2}\right) \\ \frac{d\omega}{dt} = 0 \end{cases}} \qquad \Rightarrow k = v_g \frac{\omega_{pe}^2}{2c^2|\Omega_e|} \qquad \Rightarrow \frac{dk}{dt} = -\frac{\partial}{\partial z}\frac{H}{\hbar}$$

Compare to particle: $\hbar\omega \equiv H$, $\hbar k \equiv p = -\frac{\partial H}{\partial z}$, $H = \frac{p^2}{2m}$

$$\omega = c^2k^2\frac{|\Omega_e|}{\omega_{pe}^2} \Rightarrow H = \frac{p^2}{2\left(\frac{\hbar}{2c}\frac{\omega_p^2}{|\Omega_e|}\right)} \Rightarrow m(z) = \frac{\hbar}{2c}\frac{\omega_p^2(z)}{|\Omega_e(z)|}$$

Also, $\partial_t H = 0$. ✓

b.) $\underbrace{\cancel{\partial_t}}_{0 \to \text{satiomary}} I + \underset{\partial/\partial z}{\nabla}(v_g I) = \cancel{2\gamma}^{\,0} I \Rightarrow v_g I = \text{const.}$

$$I = \frac{1}{16\pi\omega}\left[E^{*}\partial_\omega(\omega\epsilon_H)E + \frac{c^2}{\omega^2}|\vec{k}\times\vec{E}|^2\right] \qquad |\vec{k}\times\vec{E}|^2 \to \frac{c^2k^2}{\omega^2}|E_x|^2 \text{ since } k \perp E$$

$$\epsilon = \begin{pmatrix} S & -iD \\ iD & S \end{pmatrix} \qquad E = \begin{pmatrix} 1 \\ i \end{pmatrix} E_x$$

$$d_\omega(\omega D) = \partial_\omega\left(\cancel{\omega}\frac{-|\Omega_e|}{\cancel{\omega}}\frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2}\right)$$

$$= -\frac{|\Omega_e|\omega_{pe}^2}{(\omega^2 - \Omega_e^2)^2}(-2\omega) = C$$

$$\partial_\omega(\omega S) = \partial_\omega\left(\omega \frac{\omega_{pe}^2}{(\omega^2-\Omega_e^2)}\right) = \frac{\omega_{pe}^2}{(\omega^2-\Omega_e^2)} + \frac{-2\omega^2\omega_{pe}^2}{(\omega^2-\Omega_e^2)^2} = A$$

$$\Rightarrow I = \frac{|E_x|^2}{16\pi\omega}\left[\underbrace{\left(\frac{1}{\sqrt{2}}\ \frac{-i}{\sqrt{2}}\right)\begin{pmatrix}A & -iC\\ iC & A\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix}\frac{1}{\sqrt{2}}} + \frac{c^2k^2}{\omega^2}\right]$$

$$(1\ \ -i)\begin{pmatrix}A + C\\ iC + iA\end{pmatrix} = A + C + C + A = 2(A+C)$$

$$= \frac{2}{2}\left[\frac{\omega_{pe}^2}{(\omega^2-\Omega_e^2)} - \frac{2\omega^2\omega_{pe}^2}{(\omega^2-\Omega_e^2)^2} + \frac{\omega|\Omega_e|\omega_{pe}^2}{(\omega^2-\Omega_e^2)^2}\right]$$

$$= \left[\frac{\omega_{pe}^2}{(\omega^2-\Omega_e^2)^2}\left[\underbrace{\omega|\Omega_e|\cancel{\omega_{pe}^2} - 2\omega^2\cancel{\omega_{pe}^2} + \cancel{\omega_{pe}^2}\omega^2 - \cancel{\omega_{pe}^2}\Omega_e^2}_{\omega|\Omega_e| - \omega^2 - \Omega_e^2}\right]\right]$$

$$-(\Omega_e+\omega)^2 + 3\omega|\Omega_e|$$

$$\Rightarrow I = \frac{|E_x|^2}{16\pi\omega}\left[2\frac{3\omega|\Omega_e| - (\Omega_e+\omega)^2}{(\omega^2-\Omega_e^2)^2}\omega_{pe}^2 + \frac{c^2k^2}{\omega^2}\right]$$

for small ω $\Rightarrow$ $I = \frac{|E_x|^2}{16\pi\omega}\left[\frac{\omega_{pe}^2}{\Omega_e^2} + \frac{c^2k^2}{\omega^2}\right]$

$U_g = 2c^2k\frac{|\Omega_e|}{\omega_{pe}^2}$ $\qquad U_g I = \text{const}$ $\qquad c^2k^2 = \frac{\omega\,\omega_{pe}^2}{|\Omega_e|}$

$$\Rightarrow |E_x^2| = |E_x(0)|^2\left(\frac{\cancel{16}^{\,8}\pi\omega}{\cancel{2}c^2k}\left(\frac{\omega_{pe}^2}{\Omega_e^2}+\frac{c^2k^2}{\omega^2}\right)^{-1}\frac{\cancel{\omega_{pe}^2}}{\cancel{|\Omega_e|}}\right)$$

$$\cancel{\frac{|\Omega_e|^2}{\omega_{pe}^2}}\left(1+\frac{|\Omega_e|}{\omega}\right)^{-1}$$

$$\boxed{|E_x|^2 = |E_x(0)|^2\left(\frac{8\pi\omega}{c^2k}\left(1+\frac{|\Omega_e|}{\omega}\right)^{-1}\right)}$$