2014 I : Q3 MHD

(a.) To get the first R-H relation, integrate continuity equation:

$$\int_V \frac{\partial \rho}{\partial t}\, dV + \int_V \nabla\cdot(\rho\vec{v})\, dV = \int_{V_1} dV \frac{\partial \rho_1}{\partial t} + \int_{V_2} dV \frac{\partial \rho_2}{\partial t} + \int_{A_1} (\rho_1 \vec{v}_1)\cdot d\vec{A}_1 + \int_{A_2} (\rho_2 \vec{v}_2)\cdot d\vec{A}_2$$

take $V_1 \to 0$ and $V_2 \to 0$ while $A_1 \to A'$, $A_2 \to A'$ then

$$\int \rho_1 \vec{v}_1 \cdot (-\hat{n})\, dA' + \int \rho_2 \vec{v}_2 \cdot \hat{n}\, dA' = 0 \implies \boxed{\rho_2 v_2 = \rho_1 v_1}$$

Similar procedure for the others, yielding

$$\left[\left(\rho \vec{v}_1 \vec{v}_1 + \left(P + \frac{B^2}{2\mu_0}\right)\overleftrightarrow{I} - \cancel{\frac{\vec{B}\vec{B}}{\mu_0}}\right)\cdot \hat{n}\right] = 0$$

note: $[\vec{B}\cdot\vec{n}] = 0$ ($\vec{B}$ // to shock front)

↳ also true in general (normal compoent of $\vec{B}$ continuous across boundary).

$$\implies \boxed{P_2 + \frac{B_2^2}{2\mu_0} + \rho_2 v_2^2 = P_1 + \frac{B_1^2}{2\mu_0} + \rho_1 v_1^2}$$

$$\left[\left(\left(\frac{\rho v^2}{2} + \frac{\gamma P}{\gamma - 1}\right)\vec{v} + \frac{\vec{E}\times\vec{B}}{\mu_0}\right)\cdot\hat{n}\right] = 0 \qquad \vec{E} = -\vec{v}\times\vec{B}$$

$$\implies \vec{E}\times\vec{B} = -(\vec{v}\times\vec{B})\times\vec{B} \parallel \vec{v}, \hat{n}$$

$$\implies \boxed{\left(\frac{\rho_2 v_2^2}{2} + \frac{\gamma}{\gamma-1}P_2 + \frac{B_2^2}{\mu_0}\right)v_2 = \left(\frac{\rho_1 v_1^2}{2} + \frac{\gamma}{\gamma-1}P_1 + \frac{B_1^2}{\mu_0}\right)v_1}$$

For the last one, note that Maxwell's Equations require

$$[\vec{E}\times\hat{n}] = 0 \implies [(\vec{v}\times\vec{B})\times\hat{n}] = 0$$

$$\implies \boxed{v_2 B_2 = v_1 B_1}$$

(ba) $X \equiv \frac{\rho_2}{\rho_1} \Rightarrow \frac{V_2}{V_1} = \frac{1}{X}, \quad \frac{B_2}{B_1} = X$

$$P_2 = P_1 + \rho_1 V_1^2 - \rho_2 V_2^2 + \frac{1}{2\mu_0}\left(B_1^2 - B_2^2\right)$$

$V_1^2 \frac{\rho_1}{P_1}\left(1 - \frac{1}{X}\right)$

$1/\beta_1$

$$\Rightarrow \frac{P_2}{P_1} = 1 + \frac{B_1^2}{2\mu_0 P_1}\left(1 - X^2\right) + \rho_1\left(V_1^2 - X V_2^2\right)/P_1$$

$$= 1 + \frac{1}{\beta_1}\left(1 - X^2\right) + \left(1 - \frac{1}{X}\right)\gamma \frac{V_1^2}{V_S^2}$$

$V_1^2 = M^2\left(V_S^2 + V_A^2\right)$

$$= 1 + \frac{1}{\beta_1}\left(1 - X^2\right) + \gamma\left(1 - \frac{1}{X}\right) M^2 \left(1 + \frac{V_A^2}{V_S^2}\right)$$

$\frac{V_A^2}{V_S^2} = \frac{B_1^2}{\mu_0 \cancel{\rho}} \frac{\cancel{\rho}}{\gamma P_1} = \frac{2}{\gamma \beta_1}$ $\quad 2/\beta_1$

$$\boxed{\frac{P_2}{P_1} = 1 + \frac{1}{\beta_1}\left(1 - X^2\right) + \gamma\left(1 + \frac{2}{\gamma\beta_1}\right)\left(1 - \frac{1}{X}\right) M^2}$$

$$\frac{\gamma}{\gamma-1} P_2 = X\left[\frac{\gamma}{\gamma-1} P_1 + \frac{\rho_1 V_1^2}{2} + \frac{B_1^2}{\mu_0}\right] - \frac{\rho_2 V_2^2}{2} - \frac{B_2^2}{\mu_0}$$

$$\frac{P_2}{P_1} = X\left[1 + \frac{\gamma-1}{\gamma}\frac{\rho_1 V_1^2}{2P_1} + \frac{\gamma-1}{\gamma P_1}\frac{B_1^2}{\mu_0}\right] - \frac{\gamma-1}{\gamma P_1}\left(\frac{\rho_2 V_2^2}{2} + \frac{B_2^2}{\mu_0}\right)$$

$$= X + \frac{\gamma-1}{2\gamma}\left[\underbrace{\frac{\rho_1}{P_1}\left(X V_1^2 - X V_2^2\right)}_{\text{see above}}\right] + \frac{\gamma-1}{\gamma}\frac{1}{P_1\mu_0}\left(X B_1^2 - B_2^2\right)$$

$$= X + \frac{\gamma-1}{2}\left(1 + \frac{2}{\gamma\beta_1}\right)\left(X - \frac{1}{X}\right) M^2 + \frac{\gamma-1}{\gamma}\frac{B_1^2}{P_1\mu_0}\left(X - X^2\right)$$

$$\boxed{\frac{P_2}{P_1} = X + \frac{\gamma-1}{2}\left(1 + \frac{2}{\gamma\beta_1}\right)\left(X - \frac{1}{X}\right) M^2 + \frac{\gamma-1}{\gamma}\frac{2}{\beta_1}\left(X - X^2\right)}$$

$$1+\frac{1}{\beta_1}(1-X^2)+\gamma\left(1+\frac{2}{\gamma\beta_1}\right)\left(1-\frac{1}{X}\right)M^2$$

$$= X+\frac{\gamma-1}{\gamma}\frac{2}{\beta_1}(X-X^2)+\frac{\gamma-1}{2}\left(1+\frac{2}{\gamma\beta_1}\right)\left(X-\frac{1}{X}\right)M^2$$

as $M\to\infty$, drop all other terms to get

$$\gamma\underbrace{\left(1-\frac{1}{X}\right)}_{\frac{X-1}{X}}\approx\frac{\gamma-1}{2}\underbrace{\left(X-\frac{1}{X}\right)}_{\frac{X^2-1}{X}} \Rightarrow \frac{2\gamma}{\gamma-1}=\frac{X^2-1}{X-1}=\frac{\cancel{(X-1)}(X+1)}{\cancel{(X-1)}}$$

$$\Rightarrow X=\frac{2\gamma}{\gamma-1}-1=\frac{2\gamma-\gamma+1}{\gamma-1} \Rightarrow \boxed{X\to\frac{\gamma+1}{\gamma-1}}$$

(d.) $s\equiv\ln(P/\rho^\gamma)$. Require $s_2-s_1>0$

$$\ln\left(\frac{P_2}{\rho_2^\gamma}\right)-\ln\left(\frac{P_1}{\rho_1^\gamma}\right)=\ln\left(\frac{P_2\,\rho_1^\gamma}{P_1\,\rho_2^\gamma}\right)=\ln\left(\frac{P_2}{P_1}\right)+\gamma\ln\left(\frac{1}{X}\right)$$

$$\cancel{\Rightarrow \ln\left[\gamma\left(1+\frac{2}{\gamma\beta_1}\right)\left(1-\frac{1}{X}\right)M^2+\frac{1}{\beta_1}(1-X^2)+1\right]+\gamma\ln\left(\frac{1}{X}\right)>0}$$

$$\ln\left(\frac{P_2}{P_1}\right)>-\ln\left(\frac{\rho_1^\gamma}{\rho_2^\gamma}\right)=\ln\left(\frac{\rho_2^\gamma}{\rho_1^\gamma}\right) \Rightarrow \underline{\frac{P_2}{P_1}>\left(\frac{\rho_2}{\rho_1}\right)^\gamma=X^\gamma}$$

$\Rightarrow$

$$\gamma\left(1+\frac{2}{\gamma\beta_1}\right)\left(1-\frac{1}{X}\right)M^2+\frac{1}{\beta_1}(1-X^2)+1>X^\gamma$$

$\Rightarrow$ non-compressive shocks can never appear in nature

(ie $X\neq 1$)

Furthermore, we need $1<X<\frac{\gamma+1}{\gamma-1}$ to maintain

$\Delta s>0$ $\Rightarrow$ shocks requires compression.