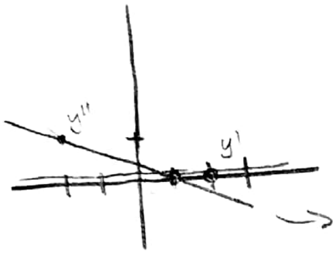
2014 I : Q 6 Math

$$\epsilon \frac{d^2y}{dx^2} + x^3 \frac{dy}{dx} - 2xy = 0 \qquad y(0) = y(1) = 1 \qquad 0 < \epsilon \ll 1$$

a.)



$$\epsilon y'' - 2xy \sim 0 \Rightarrow \epsilon x^{-2} \sim 2x$$

$$\epsilon \sim x^3 \Rightarrow \boxed{\delta \sim \epsilon^{1/3}}$$

$\rightarrow$ slope $= -3 \Rightarrow x \sim \epsilon^{1/3}$

b.) Assume layer on left then check later.

Outer layer: $x^3 y' - 2xy \sim 0$

$$\int \frac{2}{x^2}\,dx = -\frac{2}{x}$$

$$\frac{y'}{y} \sim \frac{2}{x^2} \Rightarrow y(x) \sim Ce^{-2/x}$$

$$e^{-2}e^{2} = 1$$

BC: $y(1) = 1 \Rightarrow \boxed{y(x) \sim e^{2(1 - 1/x)}}$ $\leftarrow$ layer on left!

c.) inner layer: $\epsilon y'' - 2xy \sim 0$

let $x = \epsilon^{1/3} X \Rightarrow \frac{d}{dx}\frac{d}{dx} y = \epsilon^{-2/3} Y''$

$$\Rightarrow \epsilon^{1/3} Y'' - 2\epsilon^{1/3} X Y \sim 0 \Rightarrow Y'' - 2XY = 0$$

near the matching boundary, X is large and positive

try $Y(X) \sim e^{S(X)} \Rightarrow S'' + (S')^2 - 2X \sim 0$

$$(S')^2 \sim 2X \Rightarrow \underline{S' \sim \pm\sqrt{2}\sqrt{X}} \Rightarrow S'' \sim \frac{1}{X^{1/2}} \ll X^{1/2} \checkmark$$

$$\Rightarrow S = \pm\sqrt{2}\,\frac{2}{3} X^{3/2}$$

so, to leading order in ϵ, we have

$$y(X) \sim A\exp\left(\frac{2\sqrt{2}}{3}(X)^{3/2}\right) + B\exp\left(-\frac{2\sqrt{2}}{3}(X)^{3/2}\right)$$

$y(0) = 1 \Rightarrow A + B = 1 \qquad y(X \to \infty) = 0 \Rightarrow A = 0$

$$\Rightarrow y(x) \sim \exp\left(-\frac{2\sqrt{2}}{3}\left(\frac{x}{\epsilon^{1/3}}\right)^{3/2}\right)$$

(d.) Now use $Y(X) = \int_C f(t) e^{Xt}\, dt$

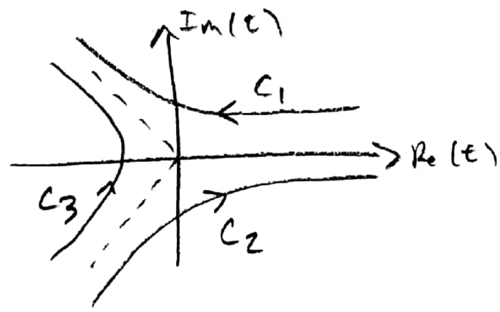
$$\Rightarrow \int_C t^2 f(t) e^{Xt}\, dt - 2\int_C X f(t) e^{Xt}\, dt$$

$$\int_C t^2 f(t) e^{Xt}\, dt + 2\int_C f'(t) e^{Xt}\, dt - \left[2 f(t) e^{Xt}\right]_C = 0$$

$$\Rightarrow t^2 f(t) + 2 f'(t) = 0$$

$$\frac{f'}{f} = -\frac{t^2}{2} \Rightarrow f(t) = e^{-t^3/6} \qquad z = X$$

$$\Rightarrow \boxed{I_k(z) = \int_{C_k} e^{zt - t^3/6}\, dt}$$ where C_k $(k = 1, 2, 3)$ is



chosen to satisfy $\left[2 e^{zt - t^3/6}\right]_C = 0 \Rightarrow$

ie $t = r e^{\frac{i 2\pi k}{3}}$

(e.) Note: $C_1 + C_2 = C_3$.

$\lim_{X \to \infty} Y_{in}(X) = \lim_{x \to 0} y_{out}(x) = 0$ $\Rightarrow$ need I_k finite as $X \to \infty$

$\Rightarrow$ choose contour C_3 since $Re[t] < 0 \Rightarrow e^{Xt} \to 0$

(f.) $y_{unif} = y_{in} + y_{out} - \cancel{y_{match}}^{\,0}$

$$y_{unif} = \int_{C_3} e^{\frac{x}{\epsilon^{1/3}} t - t^3/6}\, dt + e^{2(1 - 1/x)}$$