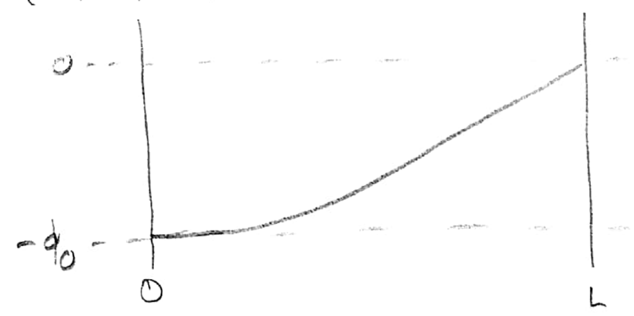
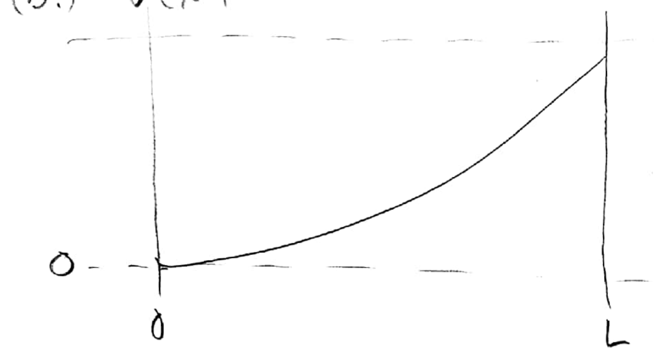
2015 I: Q1 GPP

(a.) $\phi(x)$

Note: $\left.\frac{d\phi}{dx}\right|_{x=0} = 0$



(b.) $v(x)$



(c.) $\nabla^2\phi = -\frac{ne}{\varepsilon_0}$ $\quad \cancel{\frac{\partial n}{\partial t}}^{\,0} + \nabla\cdot(n\vec{u}) = 0 \Rightarrow \vec{u}(\nabla\cdot n) + n(\nabla\cdot\vec{u}) = 0 \Rightarrow \vec{u} = \text{const}$ "divergence free"

$J = en\vec{u} \Rightarrow J = \text{const.}$

momentum eq: $m\left(\cancel{\frac{\partial v}{\partial t}}^{\,0} + (v\cdot\nabla)v\right) = q\vec{E}$

$$m v \frac{\partial v}{\partial x} = -q\frac{d\phi}{dx} \Rightarrow \frac{1}{2}mv^2 = -q\,\phi(x)$$

$$\vec{J} = en\vec{u} \Rightarrow v = \frac{J}{en} \Rightarrow q\phi(x) = -\frac{1}{2}m\frac{J^2}{(en)^2}$$

$$\Rightarrow \left(\frac{d}{dx}\frac{d\phi}{dx} = -\frac{1}{\varepsilon_0}\sqrt{\frac{-mJ^2}{2q\,\phi(x)}}\right)\frac{d\phi}{dx} \qquad \frac{d}{dx}\left(\frac{d\phi}{dx}\right)^2 = 2\frac{d\phi}{dx}\frac{d}{dx}\left(\frac{d\phi}{dx}\right)$$

$$\int dx\left[\frac{1}{2}\frac{d}{dx}\left(\frac{d\phi}{dx}\right)^2 = -\frac{1}{\varepsilon_0}\sqrt{\frac{mJ^2}{2e}}\frac{d\phi}{dx}\phi^{-1/2}\right]$$

$$\frac{1}{2}\left[\left(\frac{d\phi(x)}{dx}\right)^2 - \cancel{\left(\left.\frac{d\phi}{dx}\right|_{x=0}\right)^2}\right] = -\frac{1}{\varepsilon_0}\sqrt{\frac{mJ^2}{2e}}\int d\cancel{x}\frac{d\phi}{d\cancel{x}}\phi^{-1/2} = -\frac{1}{\varepsilon_0}\sqrt{\frac{mJ^2}{2e}}\,\cancel{2}^{\,4}\phi^{1/2}$$

0

b/c need $E|_{x=0} = 0$ to maintain const current.

$$\int\frac{d\phi}{\phi^{1/4}} = -\sqrt{\frac{4}{\varepsilon_0}}\left(\frac{mJ^2}{2e}\right)^{1/4}\int_0^L dx$$

$$\frac{4}{3}\phi^{3/4} = -\sqrt{\frac{4}{\varepsilon_0}}\left(\frac{mJ^2}{2e}\right)^{1/4}L$$

$$\cancel{\left(\frac{4}{3}\right)^4}\,\frac{4}{9}\,\phi^3\,\frac{\varepsilon_0^2}{4^2}\,\frac{1}{L^4}\,\frac{2e}{m} = J^2 \Rightarrow \boxed{J = \frac{4\varepsilon_0}{9L^2}\sqrt{\frac{2e}{m}}\,\phi^{3/2} \sim \phi^{3/2}}$$