2015 I: Q4 MHD

(a.) $k = \frac{\#}{L} = \frac{d\theta}{dz}$ $\quad B_z \sim dz, \; B_\phi \sim r\,d\theta \implies k = \frac{B_\phi}{r B_z}, \quad \vec{B} = \vec{B}(r), \; \vec{J} = \vec{J}(r)$

$$\nabla \cdot \vec{B} = 0 \implies \frac{1}{r}\frac{\partial}{\partial r}(r B_r) = 0 \implies \boxed{B_r = 0}$$

$$\vec{J} \times \vec{B} = 0 = \nabla \times \vec{B} \times \vec{B} = \left(0\,\hat{r} - \frac{\partial B_z}{\partial r}\hat{\phi} + \frac{1}{r}\frac{\partial}{\partial r}(r B_\phi)\hat{z}\right) \times \vec{B}$$

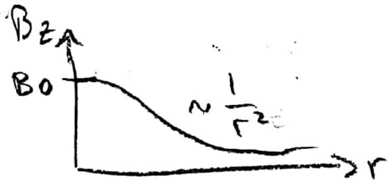
$$\implies -B_z \frac{\partial B_z}{\partial r} - \frac{B_\phi}{r}\frac{\partial}{\partial r}(\underbrace{r B_\phi}_{k r B_z}) = 0$$

$$-\cancel{B_z}\frac{\partial B_z}{\partial r} = k^2 \cancel{B_z}\left[2 r B_z + r^2 \frac{\partial B_z}{\partial r}\right]$$

$$\int \frac{dB_z}{B_z} = -2k^2 \int \frac{r\,dr}{1 + k^2 r^2} \quad \left(\begin{array}{c} u = 1 + k^2 r^2 \\ du = 2k^2 r\,dr \end{array}\right) = 2\int \frac{du}{u}.$$

$$\ln(B_z / B_0) = -\ln(1 + k^2 r^2)$$

$$\boxed{\begin{aligned} B_z &= \frac{B_0}{1 + k^2 r^2} \\ B_\phi &= \frac{k r B_0}{1 + k^2 r^2} \end{aligned}}$$



(b.) Boundary at $r = a$

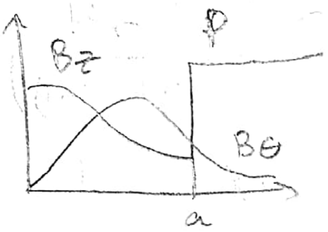
axial magnetic flux $= \int B_z \, \underbrace{dA}_{2\pi r\,dr} = 2\pi B_0 \int_0^a \frac{r}{1 + k^2 r^2}\,dr$ $\quad du = 2k^2 r\,dr, \; u = 1 + k^2 r^2, \; r\,dr = \frac{du}{2k^2}$

$$\Phi_B = \frac{\pi B_0}{k^2}\int \frac{du}{u} = \frac{\pi B_0}{k^2}\left[\ln(1 + k^2 a^2) - \cancelto{0}{\ln(1 + 0)}\right]$$

$$\boxed{\Phi_B = \frac{\pi B_0}{k^2}\ln(1 + k^2 a^2)}$$

If there is a gas of pressure P outside the flux tube, consider force balace:

$$\frac{d}{dr}\left(\frac{B_\theta^2 + B_z^2}{2\mu_0} + P\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$



$$\Rightarrow \quad p = \frac{B_\theta^2 + B_z^2}{2\mu_0}$$

$$\Rightarrow \quad p = \frac{B_0^2}{2\mu_0}\left[\frac{1+k^2a^2}{(1+k^2a^2)^2}\right] \Rightarrow \boxed{p = \frac{B_0^2}{2\mu_0}\frac{1}{1+k^2a^2}}$$

c.) Now we hold p, Φ and twist k.

$$\frac{p}{\Phi} = B_0\left(\frac{\frac{k^2}{\pi}}{2\mu_0\,(1+k^2a^2)\ln(1+k^2a^2)}\right)$$

so $$B_0 \sim \frac{p}{\Phi}\frac{(1+k^2a^2)\ln(1+k^2a^2)}{k^2}$$

$$\boxed{B_0 \sim \frac{(1+k^2a^2)\ln(1+k^2a^2)}{k^2}}$$