2016 II. Q4 Math

$y^{(4)} = xy$

a.) at large x try $y = e^{S(x)}$ (irregular singular points)

$\Rightarrow (S')^4 \sim X$ (assume $(S')^4 \gg S^{(4)}$ etc.)

$\Rightarrow S' \sim (Xe^{2\pi i k})^{1/4} = e^{i\frac{\pi}{2}k} x^{1/4} \quad k = (0,1,2,3)$

$\Rightarrow S \sim e^{i\frac{\pi}{2}k} \frac{4}{5} x^{5/4}$

$\Rightarrow \boxed{y(x) \sim \exp\left[\frac{4}{5} e^{i\frac{\pi}{2}k} x^{5/4}\right]}$

b.) Let $S(x) = S_0(x) + S_1(x)$ then $S'(x) = S_0'(x) + S_1'(x)$

$S^{(4)} + 3(S'')^2 + 4S'S^{(3)} + 6(S')^2 S'' + \underbrace{(S')^4 = X}$

plug in $S = S_0 + S_1$ X kills $(S_0')^4 + 4S_0'^3 S_1' = \cancel{X}$

$S_0' \sim x^{1/4} \quad S_0'' \sim x^{-3/4} \quad S_0''' \sim x^{-7/4} \quad S_0^{(4)} \sim x^{-11/4}$

$S^{(4)} \sim x^{-11/4} \qquad 6(S')^2 S'' \sim x^{1/2} x^{-3/4} \sim x^{-1/4}$

$3(S'')^2 \sim x^{-3/2} \qquad 3S'S^{(3)} \sim x^{1/4} x^{-7/4} \sim x^{-6/4} \sim x^{-3/2}$

Keeping only the biggest terms:

$6(S_0')^2 S_0'' + 4S_0'^3 S_1' \sim 0 \qquad S_0 = \frac{4}{5}x^{5/4} \quad S_0' = x^{1/4}$

$S_0'' = \frac{1}{4}x^{-3/4}$

$6x^{1/2}\frac{1}{4}x^{-3/4} + 4x^{3/4}S_1' \sim 0$

$S_1' \sim -\frac{3}{8}\frac{x^{-1/4}}{x^{3/4}} = -\frac{3}{8x}$

$\Rightarrow S_1 \sim -\frac{3}{8}\ln x \Rightarrow \boxed{y(x) \sim \frac{1}{x^{3/8}} \exp\left(\frac{4}{5}e^{i\frac{\pi}{2}k} x^{5/4}\right)}$

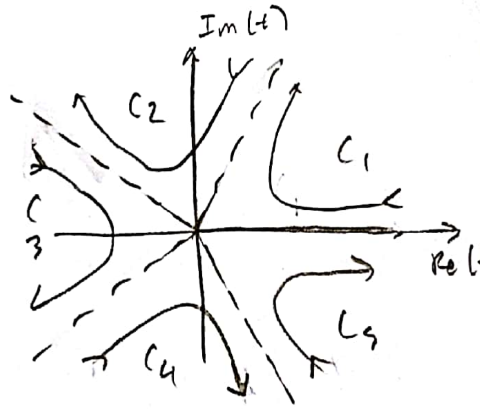
c.) $y = \int_C e^{xt} f(t)\,dt \Rightarrow \int_C f(t)\, t^4 e^{xt}\,dt = \int x f(t) e^{xt}\,dt$

$f(t) \quad e^{xt}$
$f'(t) \quad xe^{xt}$

IBP. $\Rightarrow \int_C [t^4 f(t) + f'(t)] e^{xt}\,dt - [f(t)e^{xt}]_C = 0$

$\Rightarrow f'(t) = -t^4 f(t) \Rightarrow \ln f(t) = -\frac{1}{5}t^5 \Rightarrow f(t) = e^{-t^5/5}$

$\Rightarrow \boxed{y(x) = \int_C e^{xt - \frac{1}{5}t^5}\,dt}$ where the contours are chosen such that $[\exp(xt - \frac{1}{5}t^5)]_C = 0$



$t^5 = re^{i\theta} = re^{2\pi k} \Rightarrow \arg(t) = \frac{2}{5}\pi k$

d.) These 5 contours have $[\exp(xt - \frac{1}{5}t^5)]_C = 0$

Note that $C_1 + C_2 + C_3 + C_4 = C_5$, so we only have 4 independent solutions

e.) $\phi(x,t) = xt - \frac{1}{5}t^5$

$\phi'(x,t) = x - t^4 = 0$ @ $t = \pm x^{1/4}, \pm i x^{1/4}$

$\phi''(x,t) = -4t^3$

Let $s = tx^{-1/4}$ then $\phi(x,t) = x^{5/4}\left(s - \frac{s^5}{5}\right)$ $\quad ds = x^{-1/4}\,dt$

$y(x) \sim x^{1/4} \int_C e^{x^{5/4}(s - \frac{1}{5}s^5)}\,ds$

$\phi(s) = s - \frac{1}{5}s^5$

$\phi'(s) = 1 - s^4 \Rightarrow s = \pm 1, \pm i \equiv c$

$\phi''(s) = -4s^3$

Let $\phi(s) \sim \phi(c) + (s-c)^2 \phi''(c)/2$ then

$$y(x) \sim x^{1/4} e^{x^{5/4}\phi(c)} \int_c e^{x^{5/4}(s-c)^2 \phi''(c)/2} ds$$

let $-u = x^{5/4}(s-c)^2 \phi''(c)/2 \Rightarrow s-c = \pm\sqrt{\dfrac{-2u}{x^{5/4}\phi''(c)}}$

$$-du = x^{5/4}\phi''(c)(s-c)\,ds \Rightarrow ds = -\frac{du}{x^{5/4}\phi''(c)(s-c)}$$

$$\Rightarrow y(x) \sim x^{1/4} e^{x^{5/4}\phi(c)} \int_c du \left(\frac{-1}{x^{5/4}\phi''(c)}\right)\sqrt{\frac{x^{5/4}\phi''(c)}{-2u}}\, e^{-u}$$

but $\int_c du\, e^{-u} \frac{1}{\sqrt{u}} = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\frac{2/8 + 5/8}{10/8} \to \frac{1}{x^{3/8}}$$

so $y(x) \sim e^{x^{5/4}\phi(c)} \dfrac{-x^{1/4} x^{5/8}}{x^{5/4}} \sqrt{\dfrac{1}{-2\phi''(c)}}\sqrt{\pi}$

$$y(x) \sim \pm\sqrt{\frac{\pi}{2\phi''(c)}\frac{1}{x^{3/8}}}\, e^{x^{5/4}\phi(c)} \qquad \phi''(c) = -4s^3$$

$$c = e^{i\frac{\pi}{2}k} \Rightarrow \phi(c) = \frac{4}{5} e^{i\frac{\pi}{2}k}$$

$$\Rightarrow \boxed{y(x) \sim x^{-3/8} \exp\left(\frac{4}{5} e^{i\frac{\pi}{2}k} x^{5/4}\right)}$$

(.) $y(0) = 1$, $y(+\infty) = 0$ need $e^{i\frac{\pi}{2}k} = -1 = c$ so we need to integrate over the C_3 contour, otherwise solution not bounded. Using the $c = -1$ saddle point gives

$$y(x) \sim \sqrt{\frac{\pi}{8}}\, x^{-3/8} \exp\left(-\frac{4}{5} x^{5/4}\right)$$

$y(x) \sim \int_c e^{xt - \frac{1}{5}t^5} dt$ @ $x = 0 \Rightarrow A\int_c e^{-\frac{1}{5}t^5} dt = 1 \Rightarrow A = \dfrac{1}{\int_{-\infty}^{\infty} e^{-t^5/5} dt}$

$$\Rightarrow y(x) = \frac{1}{\int_{-\infty}^{\infty} e^{t^5/5} dt} \int_{C_3} e^{xt - \frac{1}{5}t^5} dt$$