2016 I : Q1 GPP

a.) Maxwell's Equations:

Newton + Maxwell

$$\begin{cases} \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} & \nabla\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \\ \nabla\cdot\vec{E} = 4\pi\rho & \nabla\cdot\vec{B} = 0 \end{cases}$$

$$\left\{ m_j\ddot{\vec{X}}_j = e_j\left(\vec{E}(\vec{x}_j,t) + \dot{\vec{X}}_j\times\vec{B}(\vec{x}_j,t)\right)\right.$$

$$\vec{J} \equiv \sum_j e_j\dot{\vec{X}}_j(t)\,\delta_2$$

$$\rho \equiv \sum_j e_j\,\vec{X}_j(t)\,\delta_2$$

$$\delta_2 \equiv \delta(\vec{X}_j - \vec{x})$$

b.) $$\sum_j m_j\left(\frac{\partial\dot{\vec{X}}_j}{\partial t} + (\dot{\vec{X}}_j\cdot\nabla)\dot{\vec{X}}_j\right)\delta_2 = \sum_j e_j\left(\vec{E}\delta_2 + \dot{\vec{X}}_j\times B\delta_2\right)$$

$$\boxed{\sum_j m_j\frac{d}{dt}(\dot{\vec{X}}_j)\delta_2 = \sum_j e_j\left(\vec{E}(x,t)\delta_2 + \dot{\vec{X}}_j\times B(x,t)\delta_2\right)}$$

↳ not in conservative form...

c.) Time and space invariance?

d.) $$\underbrace{\frac{\partial}{\partial t}\left[\frac{\vec{E}^2+\vec{B}^2}{8\pi}\right.}_{} + \left.\sum_j\frac{m_j\dot{\vec{X}}_j^2}{2}\delta_2\right] + \nabla\cdot\left[\frac{c\,\vec{E}\times\vec{B}}{4\pi} + \sum_j\frac{m_j\dot{\vec{X}}_j^2}{2}\dot{\vec{X}}_j\delta_2\right] = 0$$

by Poynting's Theorem: $$\int dV\frac{\partial}{\partial t}\left[\frac{\vec{E}^2+\vec{B}^2}{8\pi}\right] = -\int dV\,\vec{J}\cdot\vec{E} - \int_S \frac{c}{4\pi}\vec{E}\times\vec{B}\cdot d\vec{S}$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t}\int dV\sum_j\frac{m_j\dot{\vec{X}}_j^2}{2}\delta_2}_{} - \underbrace{\int dV\,\vec{J}\cdot\vec{E}}_{\text{ohmic heating}} + \underbrace{\int dV\,\nabla\cdot\sum_j\frac{m_j\dot{\vec{X}}_j^2}{2}\dot{\vec{X}}_j\delta_2}_{} = 0$$

$\vec{\nabla}\cdot Q \to 0$ (kinetic energy flux through boundary at ∞)

$$= \int dV\sum_j m_j\dot{\vec{X}}_j\cdot\ddot{\vec{X}}_j\delta_2$$

$$= \underbrace{\int dV\sum_j\dot{\vec{X}}_j\,e_j\vec{E}\,\delta_2}_{\vec{J}\cdot\vec{E}} + \frac{1}{c}\sum_j\cancelto{0}{\dot{\vec{X}}_j\cdot\dot{\vec{X}}_j\times\vec{B}}\,\delta_2$$

$$= \int dV\,\vec{J}\cdot\vec{E} - \int dV\,\vec{J}\cdot E = 0$$

$\Rightarrow$ global energy conservation.

LOCAL:

$$\underbrace{\sum_j \dot{x}_j m_j \ddot{x}_j \delta_2}= \sum_j e_j \dot{x}_j (E(x,t))\delta_2 + \sum_j e_j \dot{x}_j \cdot \cancel{\dot{x}_j \times B(x,t)}\,\delta_2 \to 0$$

$$\sum_j \frac{d}{dt}\left[\frac{1}{2} m_j \dot{x}_j^2\right]\delta_2 = \sum_j E\cdot J$$

$$= \sum_j E \cdot \frac{c\nabla\times B - \frac{\partial E}{\partial t}}{4\pi}$$

Poynting's Theorem

$$\underbrace{\qquad\qquad} = -\frac{\partial}{\partial t}\left(\frac{E^2+B^2}{8\pi}\right) - \nabla\cdot c\frac{(E\times B)}{4\pi}$$

$$= \sum_j \frac{d}{dt}\left[\frac{1}{2} m_j \dot{x}_j^2 \delta_2\right] - \sum_j \frac{1}{2} m \dot{x}_j^2 \frac{d}{dt}(\delta_2)$$

$$\frac{d}{dt}\delta(x_j(t)-x)$$

$$= \frac{d}{dx_j}\delta(x_j(t)-x)\,\dot{x}_j$$

$$= \frac{d}{dx}\delta_2\,\dot{x}_j$$

$$= \frac{d}{dt}\sum\left[\frac{1}{2} m \dot{x}_j^2 \delta_2\right] - \sum \frac{1}{2} m \dot{x}_j^3 \frac{d}{dx}\delta_2$$

$$= \frac{d}{dt}\sum\left[\frac{1}{2} m_j \dot{x}_j^2 \delta_2\right] - \nabla \sum \frac{1}{2} m \dot{x}_j^2 \dot{x}_j \delta_2$$

$$\Longrightarrow \frac{\partial}{\partial t}\left[\frac{E^2+B^2}{8\pi} + \sum_j \frac{m_j \dot{x}_j^2}{2}\delta_2\right] + \nabla\cdot\left[\frac{cE\times B}{4\pi} + \sum_j \frac{m_j \dot{x}_j^2}{2}\dot{x}_j \delta_2\right] = 0$$

QED