2016I : Q6 MHD

(a.) $\delta W = \frac{1}{2}\int dV \left[\gamma P(\nabla\cdot\xi)^2 + (\xi\cdot\nabla P)\nabla\cdot\xi + (\xi\cdot g)\underbrace{(\nabla\cdot\rho\xi)}_{=\rho\nabla\cdot\xi + \xi\cdot\nabla\rho}\right]$

$$2\delta W = \int dV \left[\gamma P(\nabla\cdot\xi) + \xi\cdot\nabla P + \rho\xi\cdot g\right](\nabla\cdot\xi) + (\xi\cdot g)(\xi\cdot\nabla\rho)$$

Then $\frac{d(\delta W)}{d(\nabla\cdot\xi)} = \frac{1}{2}\int dV\left[2\gamma P(\nabla\cdot\xi) + \xi\cdot\nabla P + \rho\xi\cdot g\right] = 0$

$$\Rightarrow \nabla\cdot\xi = -\frac{\xi\cdot\nabla P + \rho\xi\cdot g}{2\gamma P}$$

note that $\nabla P = \rho g \Rightarrow \boxed{\nabla\cdot\vec{\xi} = -\vec{\xi}\cdot\frac{\rho\vec{g}}{\gamma P}} = -\vec{\xi}\cdot\nabla\ln P^{1/\gamma}$

(b.) For stability need $\delta W > 0$

$$\delta W = \frac{1}{2}\int dV\left[-\cancel{\vec{\xi}\cdot\rho\vec{g}} + \xi\cdot\nabla P + \cancel{\rho\xi\cdot g}\right]\nabla\cdot\vec{\xi} + (\xi\cdot \overset{g = \frac{\nabla P}{\rho}}{g})(\xi\cdot\nabla\rho)$$

$$0 < \int dV\,(\vec{\xi}\cdot\nabla P)\left(-\vec{\xi}\cdot\frac{\rho\vec{g}}{\gamma P}\right) + \frac{1}{\rho}(\xi\cdot\nabla P)(\xi\cdot\nabla\rho)$$

$$\Rightarrow \frac{1}{\rho}(\xi\cdot\nabla\rho)\cancel{(\xi\cdot\nabla P)} > \left(\xi\cdot\frac{\nabla P}{\gamma P}\right)\cancel{(\xi\cdot\nabla P)}$$

if $\xi\cdot\nabla P > 0$
" $\xi\cdot\rho g$

$$\xi\cdot\nabla\ln\rho > \frac{1}{\gamma}\xi\cdot\nabla\ln P$$

$$\Rightarrow \nabla\ln\rho^\gamma > \nabla\ln P \Rightarrow 0 > \nabla\ln\left(\frac{P}{\rho^\gamma}\right)$$

$$\Rightarrow \text{so } \boxed{\nabla\left(\frac{P}{\rho^\gamma}\right)\cdot\vec{g} < 0}$$

(c.) $PV^\gamma \sim \frac{P}{B^\gamma} = \text{const}$ but $P = B^2/2\mu_0$ so need $\boxed{\gamma = 2}$

(d.) $\rho g = \nabla P_B \Rightarrow \gamma = 2$ so $\nabla\left(\frac{P}{\rho^\gamma}\right)\cdot\vec{g} < 0$

becomes $\nabla\left(\frac{P}{\rho^2}\right)\cdot\vec{g} < 0 \Rightarrow \boxed{\nabla\left(\frac{B^2}{\rho^2}\right)\cdot\vec{g} < 0}$

(e.) $\vec{Q} \equiv \nabla\times(\vec{\xi}\times\vec{B})$

line bending = 0

$= \xi(\underbrace{\nabla\cdot B}_{0}) - B(\nabla\cdot\xi) + \cancel{(B\cdot\nabla)\xi} - (\xi\cdot\nabla)B$

$= -B(\nabla\cdot\xi) - (\xi\cdot\nabla)B$

$= +\vec{\xi}\cdot\left(\frac{\rho\vec{g}}{\gamma P}B\right) - \xi\cdot\nabla B$

~~$\sim \vec{\xi}\cdot(\vec{B}) \ll \vec{B}$ since ξ is a small pertubation? Not sure what else to do.~~

~~Line bending decreases stability since plasma can slide down field line.~~

If we formally set $Q = 0$

$\Rightarrow \nabla\cdot\xi = -\frac{\xi\cdot\nabla B}{B} = -\xi\cdot\nabla\ln B$

$\nabla\cdot\vec{\xi} = \vec{\xi}\cdot\frac{\rho\vec{g}}{\gamma P}$

$= -\xi\cdot\frac{\nabla P}{\gamma P}$

$= -\xi\cdot\nabla\ln P^{1/\gamma}$

so with $B \sim P^{1/\gamma}$, $Q \sim 0$

$P = \frac{B^2}{2\mu_0} \Rightarrow B \sim P^{1/2}$ and $\gamma = 2$

so the $|Q| \ll B$ assumption is justified.