2017 II : Q3 Math

a.) $-\nu y + ixy + 1 = 0 \quad \Rightarrow y = \dfrac{-1}{ix-\nu}$

$$\tau = \lim_{\nu \to 0} \int_{-\infty}^{\infty} J \frac{-1}{ix-\nu}\, dx \approx J i \ln x \Big|_{-\infty}^{\infty}$$

$$= iJ\left[\ln(\infty) - \ln(-\infty)\right]$$

$$= iJ\left[\cancel{\ln(\infty)} - \cancel{\ln(\infty)} - \ln(e^{i\pi})\right]$$

$$\Rightarrow \boxed{\tau = \pi J}$$

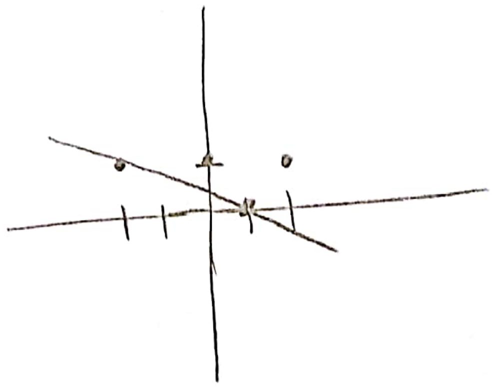
b.) $\nu \dfrac{d}{dx}\left(\operatorname{sech}(x) \dfrac{dy}{dx}\right) + ixy + 1 = 0$

The outer layer solution has $ixy + 1 = 0 \quad \Rightarrow y = \dfrac{i}{x}$

The boundary layer is in the middle (small x)

c.) $\nu \dfrac{d}{dx}\left[\dfrac{dy}{dx}\left(1 - \dfrac{x^2}{2} + \dfrac{5x^4}{24}\right)\right] + ixy = 0$

$$\nu y''\left(1 - \frac{x^2}{2} + \frac{5x^4}{24}\right) + \nu y'\left(-2x - \frac{5}{6}x^3\right) + ixy = 0$$



Inner layer: $\nu y'' + ixy = 0$

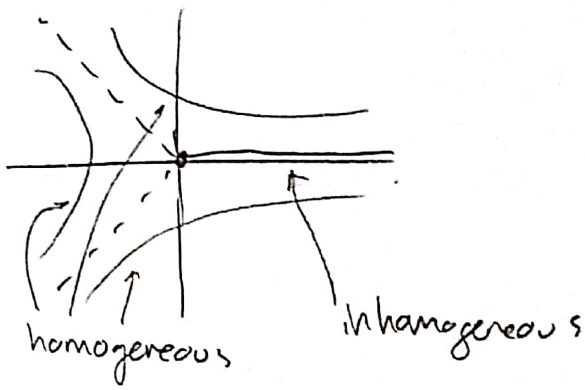
$\delta \sim \nu^{1/3}$

d.) Let $x = \nu^{1/3} X$, $y = \nu^{-1/3} Y \quad \Rightarrow \boxed{\dfrac{d^2 Y}{dX^2} + iXY = -1}$

e.) $y \sim \int_C f(t) e^{ixt}\, dt \Rightarrow \int_C -t^2 f(t) e^{ixt}\, dt + \int_C ix e^{ixt} f(t)\, dt = -1$

$$\Rightarrow \underbrace{\int_C \left(-t^2 f(t) - f'(t)\right) e^{ixt}\, dt} + \left[f(t) e^{ixt} \right]_C = -1$$

$$\Rightarrow f(t) = e^{-\frac{1}{3}t^3}$$



$$\Rightarrow \boxed{y_p(x) = \int_0^\infty e^{ixt - t^3/3}\, dt}$$

f.) $\phi(t) = ixt - t^3/3 \Rightarrow \phi(t_0) = \pm\frac{2}{3} ix\sqrt{ix} = \pm\frac{2}{3} e^{i3\pi/4} x^{3/2}$

$\phi'(t) = ix - t^2 \Rightarrow t_0 = \pm\sqrt{ix}$

$\phi''(t) = -2t \Rightarrow \phi''(t_0) = \mp 2\sqrt{ix}$

Through the saddle point : $I \sim e^{\phi(t_0)} \sim e^{\pm\frac{2}{3} e^{i3\pi/4} x^{3/2}}$

$$I \sim e^{\mp\frac{2}{3}\cos\left(\frac{\pi}{4}\right) x^{3/2}}$$

Through the end point : $I \sim \int_0^\infty e^{ixt}\, dt = \frac{1}{ix}\Big|_0^\infty \sim -\frac{1}{ix} = \frac{i}{x}$

$\frac{i}{x} \gg e^{-x^{3/2}}$ for $x \Rightarrow \infty$ so $\boxed{y_{unif} = \frac{i}{x}}$

g.) $\Upsilon = \lim_{\nu \to 0} \int_{-\infty}^{\infty} J \frac{i}{x}\, dx \Rightarrow \boxed{\Upsilon = \pi J}$

Same as krook operator!