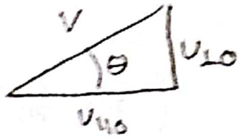
2017 II : Irreversibles Q4

(a.) Confined if $\frac{1}{2} m v_{\parallel 0}^2 + \mu B_0 \leq \mu B_1$ ie $\omega_{\parallel 0} < \omega_{\perp 0} \left(\frac{B_1}{B_0} - 1 \right)$

$$\frac{\xi^2}{1-\xi^2} = \frac{v_{\parallel 0}^2}{v_{\perp 0}^2} \leq R - 1$$



$$\xi = \frac{v_\parallel}{v} = \cos\theta$$

$$\frac{v_\parallel^2}{v_\perp^2} = \frac{\cos^2\theta}{\sin^2\theta} = \frac{\xi^2}{1-\xi^2}$$

$$\frac{1}{R-1} \leq \frac{1-\xi^2}{\xi^2} = \frac{1}{\xi^2} - 1$$

$$\xi^2 \leq \frac{1}{1+\frac{1}{R-1}} = \frac{R-1}{\cancel{R-1+1}} = 1 - \frac{1}{R}$$

so $\boxed{\xi \leq \sqrt{1 - 1/R} \equiv \xi_{crit}}$

(b.) mono-energetic source: $S = S\,\delta(v - v_0)$

$$\cancel{\left(\frac{\partial f}{\partial t}\right)_c}^{\,0} = C[f] + S\,\delta(v-v_0) = \frac{\nu(v)}{2} \frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial f_{eq}}{\partial \xi} + S\,\delta(v-v_0)$$

integrate over ξ: $-\int d\xi\, S\,\delta(v-v_0) = \int d\xi\, \frac{\nu(v)}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial f_{eq}}{\partial \xi}$

$$-\frac{2S}{\nu(v)}\delta(v-v_0)\,\xi = (1-\xi^2)\frac{\partial f_{eq}}{\partial \xi} + \underbrace{\text{const.}}$$

need BC on $f_{eq}'(\xi)$

note that $f(\xi) = f(-\xi)$ so we have $f_{eq}'(\xi) = 0$ at $\xi = 0 \Rightarrow c = 0$

$$-\int d\xi\, \frac{2S}{\nu(v)}\delta(v-v_0)\frac{\xi}{1-\xi^2} = \int d\xi\, \frac{\partial f_{eq}}{\partial \xi}$$

$u = 1-\xi^2 \quad du = -\xi\, d\xi$

$$\Rightarrow f_{eq} = +\frac{2S}{\nu(v)}\delta(v-v_0)\ln(1-\xi^2) + \underbrace{\text{const.}}$$

need BC for $f(\xi)$

note that $f(\xi_{crit}) = 0$

$$\Rightarrow \boxed{f_{eq} = \frac{2S}{\nu(v)}\delta(v-v_0)\ln\left(\frac{1-\xi^2}{1-\xi_{crit}^2}\right)}$$

(c.) The confinement time can be written as $\langle \tau_E \rangle \sim \dfrac{\#\text{ of particles}}{\langle\text{loss rate}\rangle}$

$$\text{so} \quad \tau_E = \frac{\int f_{eq}\, d\vec{v}}{-\int d\vec{v}\, C[f]} = \frac{\int f_{eq}\, d\vec{v}}{\int S\delta(v-v_c)\, d\vec{v}} \qquad \int d\vec{v} = \int_0^\infty 2\pi v^2 dv \int_{-1}^{1} d\xi$$

$$\tau_E = \frac{\cancel{2\pi} \frac{\cancel{S v_0^2}}{\nu(v_0)}}{\cancel{2\pi S v_0^2}} \int_{-\xi_c}^{\xi_c} d\xi \ln\left(\frac{1-\xi^2}{1-\xi_{crit}^2}\right)$$

$$= \frac{1}{\nu(v_0)}\left[\xi \ln\cancel{(1-\xi^2)} - 2\xi + \ln\left(\frac{1+\xi}{1-\xi}\right) - \xi\left(1-\cancel{\xi_{crit}^2}\right)\right]_{-\xi_c}^{\xi_c}$$

$$\boxed{\tau_c = \frac{2}{\nu(v_c)}\left[\ln\left(\frac{1+\xi_{crit}}{1-\xi_{crit}}\right) - 2\xi_{crit}\right]}$$

$R_m \sim 1 \Rightarrow \xi_{crit} \ll 1$

$$\Rightarrow \tau_c \sim \frac{2}{\nu(v_0)}\left[\cancel{\xi_c} - \cancel{\frac{\xi_c^2}{2}} + \frac{\xi_c^3}{3} - \cancel{2\xi_c} + \cancel{\xi_c} + \cancel{\frac{\xi_c^2}{2}} + \frac{\xi_c^3}{3}\right]$$

$$\Rightarrow \tau_c \sim \frac{4}{3}\frac{\xi_c^3}{\nu(v_c)}$$

$R_m \gg 1 \Rightarrow \xi_{crit} \sim 1 + \frac{1}{R_m}\left(-\frac{1}{2}\right) = 1 - \frac{1}{2R_m}$

$$\Rightarrow \tau_c \sim \frac{2}{\nu(v_0)}\left[\ln\left(\frac{2 - \frac{1}{2R_m}}{\frac{1}{2R_m}}\right) - 2 + \cancel{\frac{1}{R_m}}\right]$$

$$\Rightarrow \tau_c \sim \frac{2}{\nu(v_0)}\left[\ln(4R_m) - 2\right]$$

so particles are lost faster in the limit $R_m \sim 1$. This makes sense physically because $R_m \sim 1$ means $B_0 \sim B_1$ which means there is no confining field.

(d.) $\nu \sim \frac{1}{\sqrt{m}}$ so $\tau_{c,i} > \tau_{c,e}$ so electrons leak faster.

$$\Omega \frac{\partial f_1}{\partial \varphi} + \nu(v)\,\mathcal{L}[f_1] = \vec{v}\cdot\left[\frac{\vec{R}}{nT} + \left(\frac{v^2}{v_{th}^2} - \frac{5}{2}\right)\vec{\nabla}\ln T\right] f_0$$

$$q_\parallel = \int d\vec{v}\; m v^2 \langle v\rangle_\varphi \langle f\rangle_\varphi = \int d\vec{v}\; m v^2 v_\parallel \hat{b} \langle f_1\rangle_\varphi$$

$$\left\langle \Omega \frac{\partial f_1}{\partial \varphi} + \nu(v)\,\mathcal{L}[f_1] = \vec{v}\cdot\left[\frac{\vec{R}}{nT} + \left(\frac{v^2}{v_{th}^2} - \frac{5}{2}\right)\vec{\nabla}\ln T\right] f_0 \right\rangle$$

$$\nu(v)\,\mathcal{L}[\langle f_1\rangle_\varphi] = v_\parallel\left[\frac{R_\parallel}{nT} + \left(\frac{v^2}{v_{th}^2} - \frac{5}{2}\right)\nabla_\parallel \ln T\right] f_0$$

$$\langle f_1\rangle_\varphi = \sum_\ell P_\ell(\xi)\, a(\xi) \qquad v_\parallel = v P_1(\xi)$$

$$\left.\mathcal{L}[\langle f_1\rangle_\varphi] = -\sum_\ell \frac{\ell(\ell+1)}{2} P_\ell(\xi)\, a(\xi)\right\} \quad \langle f_1\rangle_\varphi = P_1(\xi)\, a_1(\xi)$$

$$\Rightarrow \langle f_1\rangle_\varphi = -\frac{f_0(v)}{\nu(v)}\, v P_1(\xi)\left[\frac{R_\parallel}{nT} + \left(\frac{v^2}{v_{th}^2} - \frac{5}{2}\right)\nabla_\parallel \ln T\right]$$

so
$$q_\parallel = \int_0^\infty 2\pi v^2 dv \int_{-1}^{1} d\xi \; -\frac{m v^2 n}{\pi^{3/2} v_{th}^3} \exp\left(-\frac{v^2}{v_{th}^2}\right) \frac{4\tau_e}{3\sqrt{\pi}} \left(\frac{v}{v_{th}}\right)^3 (v P_1(\xi))^2 \cdot [\ldots]\,\hat{b}$$

$$\int_{-1}^{+1} d\xi\; P_1(\xi) P_1(\xi) = \frac{2}{3}$$

Need to determine $R_\parallel \longrightarrow \int dv\; v_\parallel \langle f_1\rangle_\varphi = 0$

$$\Rightarrow 0 = \int v^2 dv\; e^{-v^2/v_{th}^2} v^3 v v \left[\frac{R_\parallel}{nT} + \left(\frac{v^2}{v_{th}^2} - \frac{5}{2}\right)\nabla_\parallel \ln T\right]$$

let $x = \frac{v^2}{v_{th}^2}$ then $dx \sim v\,dv \Rightarrow \int dx\; x^3 e^{-x}\left[\frac{R_\parallel}{nT} + \left(x - \frac{5}{2}\right)\nabla_\parallel \ln T\right] = 0.$

$$\Rightarrow \left[\frac{R_\parallel}{nT} - \frac{5}{2}\nabla_\parallel \ln T\right]\cancel{\Gamma(3)} = -\nabla_\parallel \ln T\, \cancel{\Gamma(4)}\; 4$$

$$\frac{R_\parallel}{nT} = -\frac{3}{2}\nabla_\parallel \ln T$$

Plug R_u back in ...

$$q_{\parallel} = (\quad) \int_0^\infty v^2 dv \; v^3 v^3 e^{-v^2/v_{th}^2} v \left[\frac{v^2}{v_{th}^2} - 4\right] \qquad dx = \frac{2v}{v_{th}^2} dv$$

$$= (\quad) \int_0^x dx \; x^4 e^{-x} (x-4) \frac{v_{th}^2}{2} v_{th}^8$$

$$= \frac{v_{th}^{10}}{2} (\quad) \left[\Gamma(5) - 4\Gamma(4)\right] = \frac{24}{2} v_{th}^{10} (\quad)$$

$$\Rightarrow \vec{q}_{\parallel} = \frac{48}{3\sqrt{\pi}} \left(\frac{2}{3}\right) \frac{2\pi}{\pi^{3/2}} mn \tau_{coll} \frac{v_{th}^{10}}{v_{th}^6} \nabla_{\parallel} \ln T \, \hat{b} \qquad v_{th}^2 = \frac{T}{m}$$

$$\boxed{\vec{q}_{\parallel} = \frac{64}{3\pi} n v_{th}^2 \tau_{coll} \nabla_{\parallel} T \, \hat{b}} \qquad m v_{th}^2 = T$$

(f.) Hydrostatic equilibrium: $\nabla \cdot \vec{q}_{\parallel} = 0$, $\quad \tau_{coll} \sim T^{3/2}$, $v_{th}^2 \sim T$

$$\nabla \cdot (T^{3/2} T \nabla_{\parallel} T) = 0$$

$$\frac{d}{dz}\left[T^{5/2} \frac{dT}{dz}\right] = 0$$

$$\int T^{5/2} dT = \int c \, dz \Rightarrow \frac{2}{7}(T - T_0)^{7/2} = cz$$

$$T = T_0 \left(\frac{7c}{2} z\right)^{2/7}$$

$$\Rightarrow \boxed{T \sim z^{2/7}}$$