2017 I: Q1 GPP

(a.) $\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{v}) = 0$

$m\left(\frac{\partial v}{\partial t} + v\cdot\nabla v\right) = qE$

1D

$n = n_0 + \tilde{n}$

$v = \cancel{v_0} + \tilde{v}$, $v_0 = 0$

Assume homogeneous, stationary $\Rightarrow \tilde{n}, \tilde{v} \sim e^{ikx - i\omega t}$

$\nabla\cdot\tilde{E} = 4\pi e(n_i - n_e)$

FT $\Rightarrow -i\omega\tilde{n}_e + n_0 ik\tilde{v}_e = 0$ $\qquad -mi\omega\tilde{v}_e = -e\tilde{E}$ $\qquad ik\tilde{E} = 4\pi e(\tilde{n}_i - \tilde{n}_e)$

$-i\omega\tilde{n}_i + n_0 ik\tilde{v}_i = 0$ $\qquad -im\omega\tilde{v}_i = e\tilde{E}$

$\Rightarrow -i\omega(\tilde{n}_i - \tilde{n}_e) + ikn_0(\tilde{v}_i - \tilde{v}_e) = 0$ $\qquad -i\omega m(\tilde{v}_i - \tilde{v}_e) = 2e\tilde{E}$

$$-\cancel{i}\omega\cancel{(\tilde{n}_i - \tilde{n}_e)} + \cancel{i}kn_0\left(\frac{2e}{-i\omega m}\,\frac{4\pi e\cancel{(\tilde{n}_i - \tilde{n}_e)}}{\cancel{ik}}\right) = 0$$

$$\boxed{\omega^2 = 2\,\frac{4\pi e^2 n_0}{m} \equiv \omega_*^2}$$

Assumptions used: stationary, homogeneous $\Rightarrow e^{ikx - i\omega t}$

- Fluid model $\Rightarrow \frac{\lambda_{mfp}}{L} \ll 1$
- cold $\left(\frac{\nabla P/n}{qE} \sim \frac{\cancel{\nabla}\cancel{n}T/\cancel{n}}{q\cancel{\nabla}\phi} \sim \frac{T}{q\phi} \ll 1\right)$

(b.) $n_e = n_0 + \tilde{n}_e \Rightarrow \frac{\partial}{\partial t}\tilde{n}_e = -n_0\frac{\partial}{\partial x}\tilde{v}_e$ $\qquad \tilde{n}_k \sim e^{ikx - i\omega t}$

$n_e(x, t=0) = n_0 + g(x) \Rightarrow \tilde{n}_e(x, t=0) = g(x)$

$$\Rightarrow \boxed{n_e(x,t) = n_0 + g(x)\cos(\omega_* t)}$$

$$|\tilde{n}_i - \tilde{n}_e| = g(x) + BC's \Rightarrow \boxed{n_i(x,t) = n_0 + g(x)\sin(\omega_* t)}$$

Now use $\frac{\partial\tilde{n}_e}{\partial t} = -n_0\frac{\partial}{\partial x}\tilde{v}_e = -g(x)\omega_*\sin(\omega_* t)$

$$\Rightarrow \tilde{V}_e = \int dx \; -\frac{\omega_*}{n_0} g(x) \sin(\omega_* t) \qquad V = \cancel{V_0}^{\,0} + \tilde{V}$$

$$BC's \Rightarrow V_e = -V_i$$

$$\boxed{\begin{aligned} V_e &= -\frac{\omega_*}{n_0} \int_0^x g(x')\,dx' \sin(\omega_* t) \\ \Rightarrow V_i &= \frac{\omega_*}{n_0} \int_0^x g(x')\,dx' \sin(\omega_* t) \end{aligned}}$$

c.) COLD: $mn_0 \frac{dv}{dt} \gg |\nabla P|$ $\qquad \omega\tilde{n} \sim n_0 \nabla \tilde{v}$

$$mn_0\omega\tilde{v} \gg T\nabla\tilde{n}$$

note: $\nabla\tilde{n} \sim k\tilde{n}$, $\nabla\tilde{v} \sim k\tilde{v}$ where $k = \left(\frac{g'}{g}\right)$

$$\Rightarrow \frac{m\omega^2}{k}\cancel{\tilde{n}} \gg Tk\tilde{n} \Rightarrow \left(\frac{g'}{g}\right)^2 \ll \frac{m\omega^2}{T}$$

$$\boxed{\frac{g'(x)}{g(x)} \ll \frac{1}{\lambda_D}} \longleftarrow \left(\frac{g'}{g}\right)^2 \ll \frac{m}{T}\frac{e^2 n_0}{m} \sim \frac{1}{\lambda_D}$$

d.) For a beam to remain the same it must not interact with the oscillations. $V_{0,beam} \equiv V_b \gg \frac{\omega}{k} \sim \omega\left(\frac{g'}{g}\right)$

$$\Rightarrow \boxed{\left(\frac{g'}{g}\right) \ll \frac{V_b}{\omega}}$$