2017 I : Q3 Exp

(a) $\tau_E = \dfrac{W}{P} = \dfrac{W}{IV}$

I : Rogowski coil
V : toroidal loop
W : diamagnetic loop + compensation coil.

(b.) Need expression for W:

$\beta_\phi = \dfrac{\langle P \rangle}{B_\phi^2/2\mu_0}$

$$W = (\text{Vol})(\text{pressure}) = 2\pi R \pi a^2 \langle P \rangle = 2\pi^2 R a^2 \beta_\phi \frac{\langle B_\phi^2 \rangle}{2\mu_0}$$

$$J \times B - \nabla P = 0 \qquad J = \frac{1}{\mu_0} \nabla \times B$$

$$\bar{J} = \frac{1}{\mu_0}\left[\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\hat{z} - \frac{\partial B_T}{\partial r}\hat{\theta}\right]$$

$$J \times B = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ & J_\theta & J_z \\ & B_\theta & B_z \end{vmatrix} = -\frac{1}{\mu_0}\left(\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)B_\theta - \frac{\partial B_T}{\partial r} B_T\right)$$

$$\Rightarrow \frac{\partial p}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\frac{B_\theta}{\mu_0} + \frac{\partial B_T}{\partial r}\frac{B_T}{\mu_0} = 0$$

$$\int_0^a r^2(\ )\,dr \Rightarrow \int_0^a r^2 \frac{dp}{dr}\,dr = \cancel{r^2 \frac{dp}{dr}\Big|_0^a} \to 0 \ (p(a)=0) - \int_0^a 2rp\,dr = -a^2\langle p \rangle$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\frac{B_\theta}{\mu_0} + \frac{\partial B_T}{\partial r}\frac{B_T}{\mu_0} = \frac{1}{2\mu_0}\frac{\partial B_T^2}{\partial r} + \frac{1}{2\mu_0}\frac{\partial B_\theta^2}{\partial r} + \frac{B_\theta^2}{\mu_0 r}$$

$$\int r^2 (B_T^2 + B_\theta^2)\,dr = \frac{a^2}{2\mu_0}(B_\theta^2 + B_\phi^2)\Big|_a - \int_0^a 2r(B_T^2 + B_\theta^2)\,dr$$

$$= \frac{a^2}{2\mu_0}(B_{\theta a}^2 + B_{\phi a}^2) - \langle B_T^2 + B_\theta^2 \rangle a^2$$

and $\displaystyle\int_0^a r\frac{B_\theta^2}{\mu_0} = \frac{1}{2\mu_0}\langle B_\theta^2\rangle a^2$

$$\Rightarrow 2\mu_0\langle p\rangle = B_{\theta a}^2 + B_{Ta}^2 - \langle B_T^2\rangle$$

NOTE : $B_\theta^2 \approx B_\theta^2(a)$

$$\Rightarrow \beta_p \equiv \frac{2\mu_0\langle p\rangle}{B_\theta^2} = 1 + \frac{B_{Ta}^2}{B_{\theta a}^2} - \frac{\langle B_T^2\rangle}{B_\theta^2(a)}$$

$$\beta \ll 1 \Rightarrow B_{Ta}^2 - \langle B_T^2\rangle \approx 2B_{Ta}(B_{Ta} - \langle B_T\rangle)$$

$$\text{so } \beta_p \approx 1 + \frac{2B_{Ta}(B_{Ta} - \langle B_T\rangle)}{B_{\theta a}^2}$$

$$\Rightarrow W \approx \frac{\pi^2 R a^2}{\mu_0}\left[B_p^2(a) + 2B_T(a)\left(B_T(a) - \langle B_T\rangle\right)\right]$$

$$B_p = \frac{\mu_0 I_p}{2\pi a} \Rightarrow I_p = \frac{2\pi a B_p(a)}{\mu_0}$$

$$\Rightarrow \boxed{\tilde{\tau}_E = \frac{\pi^2 R a^2}{\mu_0}\frac{\left[B_p^2(a) + 2B_T(a)(B_T(a) - \langle B_T\rangle)\right]}{\frac{2\pi a B_p(a)}{\mu_0}V_{loop}}}$$