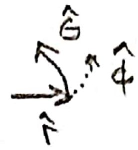
2017 I : Q4 MHD

a.) $J \times B = \nabla P$

$$J \times B \times B = \nabla P \times B \quad \Rightarrow \quad \boxed{\vec{J}_\perp = -\frac{\vec{\nabla} P \times \vec{B}}{B^2}}$$



b.) $\vec{B} = I \nabla \phi + \nabla \psi \times \nabla \phi$

$$\nabla \phi = \frac{\hat{\phi}}{R}, \quad \nabla \psi = B_\theta r \hat{r} \quad \Rightarrow \quad \nabla \psi \times \nabla \phi = \frac{r}{R} B_\theta \hat{\theta}$$

$$\vec{\nabla} P \times \vec{B} = p'(\psi) B_\theta r \hat{r} \times \left(\frac{I}{R} \hat{\phi} + \frac{r}{R} B_\theta \hat{\theta} \right)$$

$$= p'(\psi) B_\theta r \left[\frac{I}{R} \hat{\theta} - \frac{B_\theta r}{R} \hat{\phi} \right]$$

and $B^2 = \frac{I^2}{R^2} + \frac{r^2 B_\theta^2}{R^2} \sim \frac{I_0^2}{R^2}$

$$\Rightarrow \vec{J}_\perp = \frac{R^2}{I_0^2} \left[\frac{\overbrace{I}^{I_0 - I_1 \psi}}{R} \hat{\theta} - \frac{B_\theta r}{R} \hat{\phi} \right] p'(\psi) B_\theta r$$

$$\boxed{\vec{J}_\perp = (R_0 + r\cos\theta)\, p'(\psi) \frac{B_\theta r}{I_0} \left[\left(1 - \frac{I_1 \psi}{I_0}\right) \hat{\theta} - \frac{B_\theta r}{I_0} \hat{\phi} \right]}$$

c.) In cylindrical coordinates: $\nabla \cdot \vec{J}_\perp = \frac{1}{r} \frac{\partial J_{\perp\theta}}{\partial \theta} + \frac{\partial J_{\perp\phi}}{\partial \phi}$

$$\Rightarrow \nabla \cdot \vec{J}_\perp = -\sin\theta\, p'(\psi) \frac{B_\theta r}{I_0} \left(1 - \frac{I_1 \psi}{I_0}\right)$$

d.) Yes it is fine to use MHD here.

$\nabla \cdot \vec{J} = 0$ is quasineutrality but isn't violated by $\nabla \cdot J_\perp \neq 0$ because we also have $\nabla \cdot \vec{J}_\parallel \neq 0$. We have $\vec{E}$-fields all the time in MHD: Ex. Ohm's Law: $\vec{E} + \vec{v} \times B = \eta \vec{J}$.

e.) $\nabla \cdot \vec{J} = 0 \Rightarrow \nabla \cdot \vec{J}_\perp = -\nabla \cdot \vec{J}_\parallel = -B \cdot \nabla (J_\parallel / B)$

$$-\vec{B} \cdot \vec{\nabla} (J_\parallel / B) = -\sin\theta \, p'(\psi) \frac{B_\theta r}{I_0}\left(1 - \frac{I_1 \psi}{I_0}\right)$$

$$\left(\cancel{\frac{I}{R}\frac{\partial}{\partial\phi}} + \frac{B_\theta \cancel{r}}{R}\frac{1}{\cancel{r}}\frac{\partial}{\partial\theta}\right)\left(\frac{J_\parallel}{B}\right) = \sin\theta \, p'(\psi) \frac{B_\theta r}{I_0}\left(1 - \frac{I_1 \psi}{I_0}\right)$$

$$J_\parallel = B p'(\psi) \frac{B_\theta r}{I_0}\left(1 - \frac{I_1 \psi}{I_0}\right)\left[\frac{1}{B_\theta}\int_0^\theta d\theta \, (R_0 + r\cos\theta)\sin\theta\right.$$

$$J_\parallel = p'(\psi) \frac{B r}{I_0}\left(1 - \frac{I_1 \psi}{I_0}\right)\left[-R_0\cos\theta\Big|_0^\theta + \frac{r\sin^2\theta}{2}\Big|_0^\theta\right]$$

$R_0(1-\cos\theta) + \frac{r}{2}\sin^2\theta$

$$\vec{J}_\parallel = J_\parallel \frac{\vec{B}}{B} \Rightarrow \boxed{\vec{J}_\parallel = \left[R_0(1-\cos\theta) + \frac{r}{2}\sin^2\theta\right] p'(\psi)\left(1 - \frac{I_1 \psi}{I_0}\right)\frac{r}{I_0}\vec{B}}$$

f.) $\vec{E} + \vec{v} \times \vec{B} = \eta \vec{J} \Rightarrow \vec{E}_\parallel = \eta \vec{J}_\parallel$

$$\Rightarrow \boxed{\vec{E}_\parallel = \left[R_0(1-\cos\theta) + \frac{r}{2}\sin^2\theta\right] p'(\psi)\left(1 - \frac{I_1 \psi}{I_0}\right)\frac{r}{R_0}\eta \vec{B}}$$